Angle-Based Distributed Node Localizability and Localization

Liangming Chen, Zhiyun Lin, *Fellow, IEEE*, and Lihua Xie, *Fellow, IEEE*

***Abstract*—To determine the positions of free nodes in a wireless sensor network, angle-based localization approaches have been recently proposed. However, the existing angle-only localizability conditions are for the whole network, for which the existence of only one unlocalizable node implies that the whole network is unlocalizable. To efficiently obtain specific information on the free nodes' localizability, this article proposes to check each node's localizability in a distributed manner. First, we propose an algebraic condition for node localizability, based on which a distributed node localizability checking algorithm is proposed. Then, a strategy on adding triangular angle measurements to those unlocalizable nodes is proposed, under which an unlocalizable network can become localizable. Finally, by using the local information of each node's triangle degree in the network, a fully distributed localization algorithm is designed, which does not require any globally graphic information or additional internode communication. Simulation examples are provided to validate the results.**

***Index Terms*—Angle measurements, distributed localization, node localizability, triangular network, wireless sensor network.**

I. Introduction

Distributed sensor network localization has been extensively studied due to its wide applications in the Internet of Things, such as networked mobile robots [1] and large-scale sensor networks [2]. Different network localization approaches have been proposed when different sensor measurements are available, such as relative positions [3], distances [4], bearings [5], [6], and interior angles [7], [8], [9]. Two aspects have been mainly studied for every network localization problem, namely, network localizability and network localization, which aim to know under what kind of algebraic or topological conditions a given network is localizable, and propose localization algorithms for estimating the positions of free nodes, respectively [2].

The existing network localizability conditions can be mainly divided into algebraic conditions and topological conditions. To determine the localizability of sensor networks with relative position, distance, bearing, or angle measurements, algebraic conditions and topological conditions have been proposed in [3], [4], [5], [6], [7], [8], and [9]. However, even if an entire network is unlocalizable, some of its nodes can be localizable. Therefore, instead of checking an entire network's localizability, Yang and Liu [10] have introduced the novel concept of node localizability, which aims to know the localizability of each single node in the network. Henceforth, node localizability conditions have been proposed in [10], [11], [12], and [13]. Although some localizability conditions and node localizability conditions have been proposed, many of them [3], [4], [5], [6], [7], [8], [10], [13], [14] are centralized, and only a few of them [11], [12] are capable of checking localizability in a distributed manner.

On the other hand, the existing network localization algorithms can be mainly divided into continuous localization algorithms [5], [7], [14] and discrete localization algorithms [4], [6], [7], [12]. For example, continuous localization algorithms have been designed in [5] based on bearing rigidity theory. Most of the continuous localization algorithms [5], [7], [14] are distributed, in which, however, the requirement of internode continuous communication may make their implementation difficult. Compared with continuous communication, discrete localization algorithms under periodic communication are more realistic. Therefore, some discrete localization algorithms have been proposed in [4], [6], [7], and [12], which require each node to know some graphic information or need additional communication to estimate this information. Using the number of each node's associated edges in the network, a few discrete localization algorithms are fully distributed, such as the algorithms in [15].

Motivated by the aforementioned two aspects, this article focuses on checking angle-based node localizability and achieving angle-only discrete localization in a distributed manner. First, we propose an algebraic condition for the node localizability of triangular angle-constrained sensor networks. Based on this algebraic condition, a distributed node localizability checking algorithm is proposed. Then, for those unlocalizable nodes, we propose to add some triangular angle measurements such that they can also become localizable. Moreover, by using each node's triangle degree in the network, a discrete and fully distributed angle-only localization algorithm is proposed, which does not require any globally graphic information or additional internode communication.

The rest of this article is organized as follows. Section II presents the preliminaries on angle measurements and angle-based localization. Section III discusses distributed node localizability. Section IV introduces distributed localization. Simulation examples are provided in Section V. Finally, Section VI concludes this article.

II. Preliminaries

In this section, we introduce the preliminaries on angle measurements and angle-based localization.

A. Notations

Consider a planar and static network consisting of $n_a \geq 2$ anchor nodes and $n_f > 0$ free nodes. Let $\mathcal{V}_f = \{1, 2, \ldots, n_f\}$ be the set of free nodes, whose positions, denoted by $p_f = [p_1^\top, p_2^\top, \ldots, p_{n_f}^\top]^\top \in \mathbb{R}^{2n_f}$, are unknown and to be determined. Let $\mathcal{V}_a = \{n_f + 1, n_f + 2, \ldots, n\}$

Manuscript received 15 August 2023; accepted 28 October 2023. Date of publication 5 December 2023; date of current version 29 February 2024. The work of Z. Lin was supported by the National Natural Science Foundation of China under Grant 62173118. The work of L. Chen and Z. Lin was supported by the Shenzhen Key Laboratory of Control Theory and Intelligent Systems under Grant ZDSYS20220330161800001. Recommended by Senior Editor T. Iwasaki. *(Corresponding authors: Zhiyun Lin; Lihua Xie.)*

Liangming Chen and Zhiyun Lin are with the Shenzhen Key Laboratory of Control Theory and Intelligent Systems, Center for Control Science and Technology, Southern University of Science and Technology, Shenzhen 518055, China, and also with the PengCheng Laboratory, Shenzhen 518066, China (e-mail: chenlm6@sustech.edu.cn; zhiyun.lin@ieee.org).

Lihua Xie is with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: elhxie@ntu.edu.sg).

Digital Object Identifier 10.1109/TAC.2023.3339437

with $n_a + n_f = n$ be the set of anchor nodes, whose positions, denoted by $p_a = [p_{n_f+1}^\top, p_{n_f+2}^\top, \ldots, p_n^\top]^\top \in \mathbb{R}^{2n_a}$, are known by themselves. We assume that no overlapping nodes exist in $p = [p_f^\top, p_a^\top]^\top \in \mathbb{R}^{2n}$. Let $I_2, 1_n, \otimes, \lambda_{\max}(), \lambda_{\min}()$, and $\mathrm{Ker}()$ be the 2-by-2 identity matrix, $n \times 1$ column vector of all ones, the Kronecker product, the maximum eigenvalue, the minimum eigenvalue of a symmetric matrix, and the kernel of a matrix, respectively. Denote by $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ the 2-D rotation matrix with rotation angle θ. For a matrix $Q \in \mathbb{R}^{m\times n}$, let $[Q]_{i*} \in \mathbb{R}^{1\times n}$ and $[Q]_{*i} \in \mathbb{R}^{m\times 1}$ be the ith row and the ith column of Q, respectively.

B. Angle Measurements and Angle-Induced Linear Equations

Define node i's interior angle measurement $\alpha_{mij} \in [0, 2\pi)$ with respect to nodes $m, j \in \mathcal{V}_a \cup \mathcal{V}_f$ under the counterclockwise direction as [7]

$$\alpha_{mij} := \begin{cases} \arccos(b_{ij}^\top b_{im}), & \text{if } b_{ij}^\top R(\frac{\pi}{2}) b_{im} \geq 0 \\ 2\pi - \arccos(b_{ij}^\top b_{im}), & \text{otherwise} \end{cases} \tag{1}$$

where $b_{ij} := \frac{p_j - p_i}{\|p_j - p_i\|}$ is the bearing unit vector starting from p_i and pointing toward p_j. According to [7], an angle-induced linear equation in $\triangle ijm$ is written by

$$f_i^{\triangle ijm}(\alpha, p) = A_i^{\triangle ijm}(\alpha) p_i + A_j^{\triangle ijm}(\alpha) p_j + A_m^{\triangle ijm}(\alpha) p_k = 0 \tag{2}$$

where the coefficient matrices $A_i^{\triangle ijm}(\alpha) = -A_j^{\triangle ijm}(\alpha) - A_m^{\triangle ijm}(\alpha)$, $A_j^{\triangle ijm}(\alpha) = \sin\alpha_{ijm} R^\top(\alpha_{mij}) \in \mathbb{R}^{2\times 2}$, and $A_m^{\triangle ijm}(\alpha) = -\sin\alpha_{jmi} I_2 \in \mathbb{R}^{2\times 2}$ are only related to the measured interior angles $\alpha_{jmi}, \alpha_{ijm}$, and α_{mij}[7]. Since the collinearity of p_i, p_j, and p_m will degrade the linear equation (2), we require each three neighboring nodes to be noncollinear.

Now, we use triangular angularities [7] to describe triangular networks with triangular angle constraints, which are the networks we are interested in in this article. For the vertex set $\mathcal{V} = \{1, 2, \ldots, n\} = \mathcal{V}_f \cup \mathcal{V}_a$, define a three-vertex *triplet* (i, j, m) to describe the angle constraint α_{ijm}. Then, we define $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V} \times \mathcal{V} = \{(i, j, m), i, j, m \in \mathcal{V}, i \neq j \neq m\}$ as an angle set. Assume that the notation of the triplet (i, j, k) is equivalent to (k, j, i). Then, the combination of the vertex set $\mathcal{V}$, the angle set $\mathcal{A}$, and the embedding $p \in \mathbb{R}^{2n}$ is called an *angularity*, which we denote by $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$. We say $\mathcal{A}$ is a *triangular angle set* if for every $(i_1, j_1, m_1) \in \mathcal{A}$, there also exists $\{(j_1, m_1, i_1), (m_1, i_1, j_1)\} \subset \mathcal{A}$. The number of triangles in $\mathbb{A}$ is denoted by $n_{\mathcal{A}}^\triangle \in \mathbb{N}^+$. We say that $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is a *triangular angularity* if $\mathcal{A}$ is a triangular angle set. If $(i, j, m) \in \mathcal{A}$, then $\{j, m\} \in \mathcal{N}_i, \{i, m\} \in \mathcal{N}_j, \{i, j\} \in \mathcal{N}_m$, where $\mathcal{N}_i$ represents node i's neighbor set. For each $i \in \mathcal{V}_f$, there exists at least one triplet in $\mathcal{A}$ that is associated with i. Let $d_i \in \mathbb{N}^+$ be node i's triangle degree, which is the number of node i's associated triangles in triangular angularity $\mathbb{A}$.

Writing all the angle-induced linear equations (2) from a triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ into a compact form yields $M_{\mathcal{A}}(\alpha) p = 0$, where $M_{\mathcal{A}}(\alpha) \in \mathbb{R}^{2n_{\mathcal{A}}^\triangle \times 2n}$, $\alpha = [\ldots, \alpha_{ijm}, \ldots]^\top \in \mathbb{R}^{3n_{\mathcal{A}}^\triangle}$, $\forall (i, j, m) \in \mathcal{A}$, and the row (respectively, column) blocks of $M_{\mathcal{A}}(\alpha)$ are indexed by the triangles in $\mathcal{A}$ (respectively, the vertices in $\mathcal{V}$) [7]. Here, α represents $\alpha(p)$, i.e., the angles in α are calculated under the nodes' embedding p.

C. Angle-Only Localizability and Localization

Partitioning $M_{\mathcal{A}}(\alpha) = [M_{\mathcal{A}}^f, M_{\mathcal{A}}^a]$ into the free nodes' part $M_{\mathcal{A}}^f \in \mathbb{R}^{2n_{\mathcal{A}}^\triangle \times 2n_f}$ and the anchor nodes' part $M_{\mathcal{A}}^a \in \mathbb{R}^{2n_{\mathcal{A}}^\triangle \times 2n_a}$, we define a matrix $D(\alpha) \in \mathbb{R}^{2n\times 2n}$ as

$$D(\alpha) = M_{\mathcal{A}}^\top(\alpha) M_{\mathcal{A}}(\alpha) = \begin{bmatrix} D_{ff} & D_{fa} \\ D_{af} & D_{aa} \end{bmatrix} \tag{3}$$

where $D_{aa} = (M_{\mathcal{A}}^a)^\top M_{\mathcal{A}}^a \in \mathbb{R}^{2n_a\times 2n_a}$, $D_{fa} = (M_{\mathcal{A}}^f)^\top M_{\mathcal{A}}^a \in \mathbb{R}^{2n_f\times 2n_l}$, and $D_{ff} = (M_{\mathcal{A}}^f)^\top M_{\mathcal{A}}^f \in \mathbb{R}^{2n_f\times 2n_f}$. The aim of network localizability is to investigate whether free nodes' positions p_f can all be uniquely determined by anchor nodes' positions p_a and angle measurements α. Now, we present our previous result on angle-only localizability.

Lemma 1 (see [7]): For a triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$: 1) $\mathbb{A}$ is localizable if and only if D_{ff} is nonsingular; 2) if the network is localizable, then p_f can be uniquely calculated by $p_f = -D_{ff}^{-1} D_{fa} p_a$; and 3) if $n_a = 2$, then D_{ff} is nonsingular if and only if $\mathrm{Rank}(M_{\mathcal{A}}(\alpha)) = 2n - 4$.

The aim of discrete localization is to design $\hat{p}_f[k+1] = f(\hat{p}_f[k], \alpha, p_a)$ such that $\hat{p}_f[k] \to p_f$ as $k \to \infty$. According to [6] and [7], a gradient-descent localization law for the free nodes can be designed as

$$\hat{p}_f[k+1] = \hat{p}_f[k] - h D_{ff} \hat{p}_f[k] - h D_{fa} p_a \tag{4}$$

whose component form for each free node $i \in \mathcal{V}_f$ is

$$\hat{p}_i[k+1] = \hat{p}_i[k] - h(F_i^{\triangle ij_1m_1} + F_i^{\triangle j_2im_2} + F_i^{\triangle j_3m_3i}) \tag{5}$$

where $F_i^{\triangle ij_1m_1} = \sum_{(i,j_1,m_1)\in\bar{\mathcal{A}}} (A_i^{\triangle ij_1m_1})^\top f_i^{\triangle ij_1m_1}(\alpha, \hat{p}[k])$, $\bar{\mathcal{A}} \subset \mathcal{A}$, $|\bar{\mathcal{A}}| = n_{\mathcal{A}}^\triangle$ such that if $(i, j, m) \in \bar{\mathcal{A}}$, then $(j, m, i) \notin \bar{\mathcal{A}}, (m, i, j) \notin \bar{\mathcal{A}}$, and $\hat{p}_j(t) = p_j$ for $\forall j \in \mathcal{V}_a$, $h > 0$ is the sampling period, $f_i^{\triangle j_2im_2}(\alpha, \hat{p}[k]) = A_{j_2}^{\triangle j_2im_2}(\alpha)\hat{p}_{j_2}[k] + A_i^{\triangle j_2im_2}(\alpha)\hat{p}_i[k] + A_{m_2}^{\triangle j_2im_2}(\alpha)\hat{p}_{m_2}[k]$, and $f_i^{\triangle j_3m_3i}(\alpha, \hat{p}[k]) = A_{j_3}^{\triangle j_3m_3i}(\alpha)\hat{p}_{j_3}[k] + A_{m_3}^{\triangle j_3m_3i}(\alpha)\hat{p}_{m_3}[k] + A_i^{\triangle j_3m_3i}(\alpha)\hat{p}_i[k]$. Note that (5) can be implemented by using node i's angle measurement to obtain α_{jim} and communication with neighbor j to obtain α_{mji} and communication with neighbors j, m to obtain $\hat{p}_j[k], \hat{p}_m[k]$, where $j \in \{j_1, j_2, j_3\}, m \in \{m_1, m_2, m_3\}$. Clearly, the measurement topology among the free nodes is described by $\mathcal{A}$. To guarantee the convergence of $\tilde{p}_f[k] = \hat{p}_f[k] - p_f$ under (5) for localizable triangular angularities, the sampling period h should satisfy [7]

$$h < 2\min_{i=1,\ldots,2n_f} \lambda_i^{-1}(D_{ff}) = 2\lambda_{\max}^{-1}(D_{ff}). \tag{6}$$

Remark 1: The localizability condition given in Lemma 1 is centralized since $D_{ff}(\alpha)$ is related to all the nodes' angle measurements. Also, the localization law (5) is not fully distributed since the condition (6) is related to the globally graphic information $\lambda_{\max}(D_{ff})$. These two aspects are the motivation of this work, and the aim is to make them distributed.

III. Distributed Node Localizability

In this section, we first introduce a checking condition on node localizability, then develop a distributed algorithm to check each node's localizability, and finally discuss the improvement of unlocalizable networks' localizability.

A. Node Localizability Condition

Denote by $e_i \in \mathbb{R}^{2n_f}$ the natural basis of $\mathbb{R}^{2n_f}$, with the ith entry of e_i being 1 and the other $(2n_f - 1)$ entries being 0. Based on [10], we introduce the definition of node localizability for sensor networks with angle measurements.

Definition 1: A free node $i \in \mathcal{V}_f$ is said to be localizable in angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ if p_i is uniquely determined by the given angle constrains in $\mathcal{A}$ and the anchors' positions p_a.

For a triangular $\mathbb{A}$, one has the following conclusion.

Lemma 2: For a triangular angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$, if node $i\in\mathcal{V}_f$ is localizable and

$$D_{ff}[(p_1+x_1)^\top,\ldots,(p_i+x_i)^\top,\ldots,(p_{n_f}+x_{n_f})^\top]^\top + D_{fa}p_a = 0 \quad (7)$$

where $x_j\in\mathbb{R}^2, j=1,\ldots,n_f$, then x_i must be zero.

Proof: Suppose on the contrary that $x_i\neq 0$. On the one hand, $D_{fa}p_a + D_{ff}p_f = 0$ always holds according to (2) and (3). Therefore, the configuration $[p_1^\top,\ldots,p_i^\top,\ldots,p_{n_f}^\top,p_a^\top]^\top\in\mathbb{R}^{2n}$ satisfies all the given angle constraints in $\mathcal{A}$ and the anchors' positions p_a. On the other hand, (7) implies that the configuration $[(p_1+x_1)^\top,\ldots,(p_i+x_i)^\top,\ldots,(p_{n_f}+x_{n_f})^\top,p_a^\top]^\top\in\mathbb{R}^{2n}$ also satisfies the given angle constraints in $\mathcal{A}$ and the anchors' positions p_a. Since $p_i+x_i\neq p_i$, there are two different solutions for the free node i's position, which contradicts to the assumption that node i is localizable. Hence, $x_i=0$. □

Theorem 1: A free node $i\in\mathcal{V}_f$ is localizable in triangular angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ if and only if $e_{2i}\perp \mathrm{Ker}(D_{ff})$ and $e_{2i-1}\perp\mathrm{Ker}(D_{ff})$.

Proof. Sufficiency: Suppose on the contrary that node i is unlocalizable. Then, there must exist at least one another position $p_i'\in\mathbb{R}^2, p_i'\neq p_i$ for node i, which satisfies the given angle constraints in $\mathcal{A}$ and anchors' positions p_a. That is to say, (7) holds for $x_i=p_i'-p_i\neq 0$ and $x_j=p_j-p_j=0$, for j $\in\mathcal{V}\backslash$i. Then, at least one of $e_{2i}^\top[x_1,\ldots,x_i^\top,\ldots,x_{n_f}^\top]^\top\neq 0$ and $e_{2i-1}^\top[x_1,\ldots,x_i^\top,\ldots,x_{n_f}^\top]^\top\neq 0$ holds. The above conclusion, together with the fact $[x_1^\top,\ldots,x_i^\top,\ldots,x_{n_f}^\top]^\top\in\mathrm{Ker}(\mathrm{D_{ff}})$, implies a contradiction to the assumption that $e_{2i}\perp\mathrm{Ker}(D_{ff})$ and $e_{2i-1}\perp\mathrm{Ker}(D_{ff})$. Therefore, node i is localizable.

Necessity: According to (2) and (3), $D_{fa}p_a+D_{ff}p_f=0$ holds no matter whether the network is localizable. Define the kernel of matrix D_{ff} as $\mathrm{Ker}(D_{ff}) := [x_1^\top,\ldots,x_i^\top,\ldots,x_{n_f}^\top]^\top\in\mathbb{R}^{2n_f}$. It follows that $D_{fa}p_a+D_{ff}p_f+D_{ff}[x_1^\top,\ldots,x_i^\top,\ldots,x_{n_f}^\top]^\top=0$, which is the same as (7). Then, according to Lemma 2, when i is localizable, $x_i=0$ holds. It follows that $e_{2i}^\top[x_1^\top,\ldots,x_i,\ldots,x_{n_f}^\top]^\top=0$ and $e_{2i-1}^\top[x_1^\top,\ldots,x_i,\ldots,x_{n_f}^\top]^\top=0$, i.e., $e_{2i}\perp\mathrm{Ker}(D_{ff})$ and $e_{2i-1}\perp\mathrm{Ker}(D_{ff})$. □

To execute the condition in Theorem 1, an order needs to be assigned to the nodes, which can be done by preassigning each sensor with an identity or using a distributed algorithm [16] to identify the nodes.

B. Structural Properties of Matrix D_{ff}

We first give an example on D_{ff} to show the structural properties of D_{ff}.

Example 1: Take a triangular angularity $\mathbb{A}(\mathcal{V}_f\cup\mathcal{V}_a,\mathcal{A},p)$ with $\mathcal{V}_f=\{1,2,3\}$ and $\mathcal{V}_a=\{4,5,6\}$ as an example. If $\bar{\mathcal{A}}=\{(1,4,5),(1,2,4),(1,2,3)\}$, then $D_{ff}\in\mathbb{R}^{6\times 6}$ is

$$D_{ff}=(M_{\mathcal{A}}^f)^\top M_{\mathcal{A}}^f$$
$$=\begin{bmatrix}(A_1^{145})^2+(A_1^{124})^2+(A_1^{123})^2 & (A_1^{124})^\top A_2^{124}+(A_1^{123})^\top A_2^{123} & (A_1^{123})^\top A_3^{123}\\ (A_2^{124})^\top A_1^{124}+(A_2^{123})^\top A_1^{123} & (A_2^{124})^2+(A_2^{123})^2 & (A_2^{123})^\top A_3^{123}\\ (A_3^{123})^\top A_1^{123} & (A_3^{123})^\top A_2^{123} & (A_3^{123})^2\end{bmatrix} \quad (8)$$

where $A_s^{ijk}=A_s^{\triangle ijk}$ and $(A_s^{ijk})^2=(A_s^{\triangle ijk})^\top A_s^{\triangle ijk}$ for $(i,j,k)\in\bar{\mathcal{A}}$ and $s\in\{i,j,k\}$. □

To introduce the related properties of the matrix D_{ff}, we define $D_{ff}[i,j]$ as the 2-by-2 block of the matrix D_{ff}'s $(2i-1)$th to $(2i)$th rows and $(2j-1)$th to $(2j)$th columns, where $i,j\in\mathcal{V}_f$.

Lemma 3: The matrix $D_{ff}\in\mathbb{R}^{2n_f\times 2n_f}$ satisfies the following statements.

i) The diagonal blocks of D_{ff} can be described by

$$D_{ff}[i,i]=\sum_{(i,j_2,m_2)\in\mathcal{A}}(A_i^{\triangle i_sj_sm_s})^\top A_i^{\triangle i_sj_sm_s} \quad (9)$$

where $\{i,j_2,m_2\}=\{i_s,j_s,m_s\}$[1], $(i_s,j_s,m_s)\in\bar{\mathcal{A}}$, and $\{j_2,m_2\}\subset\mathcal{V}_f\cup\mathcal{V}_a$. The off-diagonal blocks of D_{ff} can be described by

$$D_{ff}[i,j]=\sum_{(i,j,m_1)\in\mathcal{A}}(A_i^{\triangle i_sj_sm_s})^\top A_j^{\triangle i_sj_sm_s} \quad (10)$$

where $i\neq j\neq m_1$, $m_1\in\mathcal{V}_f\cup\mathcal{V}_a$, and $\{i,j,m_1\}=\{i_s,j_s,m_s\}$, $(i_s,j_s,m_s)\in\bar{\mathcal{A}}$.

ii) Each component $(A_i^{\triangle i_sj_sm_s})^\top A_i^{\triangle i_sj_sm_s}\in\mathbb{R}^{2\times 2}$ in (9) can be written as

$$(A_i^{\triangle i_sj_sm_s})^\top A_i^{\triangle i_sj_sm_s}=\beta_1^{\triangle i_sj_sm_s}I_2 \quad (11)$$

where $\beta_1^{\triangle i_sj_sm_s}\in(0,1)$ for $\forall i\in\{i_s,j_s,m_s\}$.

iii) Each component $(A_i^{\triangle i_sj_sm_s})^\top A_j^{\triangle i_sj_sm_s}\in\mathbb{R}^{2\times 2}$ in (10) can be written as

$$(A_i^{\triangle i_sj_sm_s})^\top A_j^{\triangle i_sj_sm_s}=\begin{bmatrix}\beta_2^{\triangle i_sj_sm_s} & \beta_3^{\triangle i_sj_sm_s}\\ -\beta_3^{\triangle i_sj_sm_s} & \beta_2^{\triangle i_sj_sm_s}\end{bmatrix} \quad (12)$$

where $|\beta_2^{\triangle i_sj_sm_s}|<1$, $|\beta_3^{\triangle i_sj_sm_s}|<1$, and $|\beta_2^{\triangle i_sj_sm_s}|+|\beta_3^{\triangle i_sj_sm_s}|<\sqrt{2}$. □

Proof: Statement (i) can be obtained by following the definition of $R_{\mathcal{A}}^f$. To prove statement (ii), considering the first case $i=i_s$, according to the definition of coefficient matrices in (2), one has

$$(A_{i_s}^{i_sj_sm_s})^\top A_{i_s}^{i_sj_sm_s}=(\sin^2\alpha_{j_sm_si_s}+\sin^2\alpha_{i_sj_sm_s}-\varepsilon_1)I_2$$

where $\varepsilon_1 := 2\sin\alpha_{j_sm_si_s}\sin\alpha_{i_sj_sm_s}\cos\alpha_{m_si_sj_s}$ and we used the fact that $R(\theta)$ is a skew-symmetric matrix. Using product-to-sum trigonometric formulas, one has

$$\varepsilon_1=\sin\alpha_{j_sm_si_s}[\sin\alpha_{j_sm_si_s}+\sin(\alpha_{i_sj_sm_s}-\alpha_{m_si_sj_s})]$$
$$=\sin^2\alpha_{j_sm_si_s}+\sin(\alpha_{i_sj_sm_s}+\alpha_{m_si_sj_s})\sin(\alpha_{i_sj_sm_s}-\alpha_{m_si_sj_s})$$
$$=\sin^2\alpha_{j_sm_si_s}+[(\cos 2\alpha_{m_si_sj_s})-\cos 2\alpha_{i_sj_sm_s}]/2$$
$$=\sin^2\alpha_{j_sm_si_s}+\sin^2\alpha_{i_sj_sm_s}-\sin^2\alpha_{m_si_sj_s}.$$

It follows that $(A_{i_s}^{i_sj_sm_s})^\top A_{i_s}^{i_sj_sm_s}=\sin^2\alpha_{m_si_sj_s}I_2$. For the remaining cases $i=j_s$ and $i=m_s$, using (2), one directly has $(A_{j_s}^{\triangle i_sj_sm_s})^\top A_{j_s}^{\triangle i_sj_sm_s}=\sin^2\alpha_{i_sj_sm_s}I_2$ and $(A_{m_s}^{\triangle i_sj_sm_s})^\top A_{m_s}^{\triangle i_sj_sm_s}=\sin^2\alpha_{j_sm_si_s}I_2$.

To prove statement (iii), considering the first case $i=i_s$ and $j=j_s$, according to (2), one has

$$(A_{i_s}^{\triangle i_sj_sm_s})^\top A_{j_s}^{\triangle i_sj_sm_s}$$
$$=(\sin\alpha_{j_sm_si_s}I_2-\sin\alpha_{i_sj_sm_s}R(\alpha_{m_si_sj_s}))\times\sin\alpha_{i_sj_sm_s}R^\top(\alpha_{m_si_sj_s})$$
$$=\begin{bmatrix}-\cos\alpha_{j_sm_si_s}\sin\alpha_{m_si_sj_s} & -\sin\alpha_{j_sm_si_s}\sin\alpha_{m_si_sj_s}\\ \sin\alpha_{j_sm_si_s}\sin\alpha_{m_si_sj_s} & -\cos\alpha_{j_sm_si_s}\sin\alpha_{m_si_sj_s}\end{bmatrix}\times\sin\alpha_{i_sj_sm_s}$$

[1]The reason of using i_s, j_s, and m_s here is that the triangle formed by i_s, j_s, and m_s is one of $\triangle ij_2m_2$, $\triangle j_2im_2$, and $\triangle im_2j_2$, which come with different coefficient matrices when they are used to form the linear equation (2).

which implies the conclusion. Considering the second case $i = i_s, j = m_s$, one has

$$\begin{aligned}(A_{i_s}^{\triangle i_s j_s m_s})^\top A_{m_s}^{\triangle i_s j_s m_s} &= \sin\alpha_{j_s m_s i_s}\sin\alpha_{i_s j_s m_s} R(\alpha_{m_s i_s j_s}) - \sin^2\alpha_{j_s m_s i_s} I_2 \\ &= -\sin\alpha_{j_s m_s i_s} \\ &\times \begin{bmatrix}\cos\alpha_{i_s j_s m_s}\sin\alpha_{m_s i_s j_s} & \sin\alpha_{i_s j_s m_s}\sin\alpha_{m_s i_s j_s} \\ -\sin\alpha_{i_s j_s m_s}\sin\alpha_{m_s i_s j_s} & \cos\alpha_{i_s j_s m_s}\sin\alpha_{m_s i_s j_s}\end{bmatrix}\end{aligned}$$

which implies the conclusion. For the remaining case $i = j_s, j = m_s$, one has the conclusion straightforwardly. □

Now, we present a lemma on D_{ff}'s eigenvalues and eigenvectors, which are important for developing distributed checking algorithms for node localizability. Define $v_i^l(D_{ff})$ and $v_i^r(D_{ff})$ as D_{ff}'s left and right eigenvectors corresponding to $\lambda_i(D_{ff})$, respectively, where $i = 1, \ldots, 2n_f$.

Lemma 4: For any $\gamma \in \mathbb{R}^+$, $\bar{D}_{ff} := D_{ff} + \gamma I_{2n_f}$ is a nonsingular matrix. Moreover

$$\begin{aligned}&\lambda_i(D_{ff}) = \lambda_i^{-1}(\bar{D}_{ff}^{-1}) - \gamma,\ i = 1, \ldots, 2n_f \\ &v_i^l(D_{ff}) = v_i^l(\bar{D}_{ff}^{-1}),\ v_i^r(D_{ff}) = v_i^r(\bar{D}_{ff}^{-1}).\end{aligned} \tag{13}$$

Proof: Since D_{ff} is a positive-semidefinite matrix, one has that $\bar{D}_{ff}$ is a nonsingular matrix. The proof of (13) follows the same line as [17, Lemma 1]. □

C. Distributed Checking Algorithm for Node Localizability

To check triangular angularities' node localizability in a distributed manner, according to Theorem 1, we need to check the conditions $e_{2i} \perp \mathrm{Ker}(D_{ff})$ and $e_{2i-1} \perp \mathrm{Ker}(D_{ff})$ in a distributed manner. According to [12, Sec. III.C], these two conditions can be verified by using the information of the matrix D_{ff}'s eigenvalues, and the $(2i-1)$th to $(2i)$th components of D_{ff}'s eigenvectors. Note that many distributed algorithms have been proposed to estimate the eigenvalues and eigenvectors of matrices associated with graphs [17], [18], [19]. Inspired by [17], [20], and Lemma 4, we aim to develop a distributed algorithm for each free node i such that it can obtain the information of D_{ff}'s eigenvalues and eigenvectors. Before the development of the algorithm, we need to introduce the communication topology for the free nodes.

1) Communication Topology for the Free Nodes: We define two communication graphs $\bar{\mathcal{G}}_f$ and $\mathcal{G}_f$, which will be used for checking node localizability and distributed localization, respectively. According to Sections II-B and II-C, the communication topology among the free nodes for distributed localization can be described by an undirected graph $\mathcal{G}_f(\mathcal{V}_f, \mathcal{E}_f)$, where the edge set $\mathcal{E}_f$ is

$$\mathcal{E}_f := \{(i,j) | i \in \mathcal{V}_f, j \in \mathcal{V}_f, (i,j,m) \in \mathcal{A}, m \in \mathcal{V}_a \cup \mathcal{V}_f\}.$$

However, graph $\mathcal{G}_f$ is inadequate to describe the communication topology for checking node localizability because D_{ff}'s block describing the communication relation between every two neighboring nodes consists of a 2-by-2 matrix instead of a scalar. To make preparations for checking node localizability, we assume that each free node i has two virtual nodes i_{-1} and i_{-2}. The communication topology among these virtual nodes is defined as an undirected graph $\bar{\mathcal{G}}_f(\bar{\mathcal{V}}_f, \bar{\mathcal{E}}_f)$, where

$$\begin{aligned}\bar{\mathcal{V}}_f &= \{1_{-1}, 1_{-2}, \ldots\ldots, (n_f)_{-1}, (n_f)_{-2}\} \\ \bar{\mathcal{E}}_f &= \{(i_{-m}, j_{-s}) |\, \{i,j\} \subseteq \mathcal{V}_f, m \in \{1,2\}, s \in \{1,2\}, i_{-m} \neq j_{-s}, \\ &\quad i = j \,\text{or}\, (i,j) \in \mathcal{E}_f\}.\end{aligned}$$

Intuitively, if there is an edge (i,j) in $\mathcal{E}_f$, then all six possible edges among i_{-1}, i_{-2}, j_{-1}, and j_{-2} will be included in $\bar{\mathcal{E}}_f$. An example of $\bar{\mathcal{G}}_f(\bar{\mathcal{V}}_f, \bar{\mathcal{E}}_f)$ with four free nodes is provided in Fig. 1. Obviously, $|\bar{\mathcal{V}}_f| = 2|\mathcal{V}_f|$ and $|\bar{\mathcal{E}}_f| = 4|\mathcal{E}_f| + |\mathcal{V}_f|$.

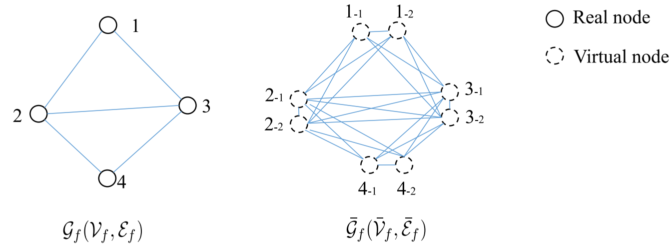

Fig. 1. Relationship between $\mathcal{G}_f$ and $\bar{\mathcal{G}}_f$.

Note that $\mathbb{A}$ being localizable does not imply that $\mathcal{G}_f$ is connected (see, e.g., [7, Fig. 2]). However, $\mathcal{G}_f$ being connected indeed implies that $\bar{\mathcal{G}}_f$ is also connected. Therefore, we present the following assumption.

Assumption 1: The graph $\mathcal{G}_f$ is connected.

Assumption 1 does not hold if the free nodes are separated by the anchor nodes into several clusters or the topology of the entire network is not connected. For the former case, the node localizability checking algorithm can still be executed in each cluster under the assumption that those free nodes in each cluster are connected.

2) Distributed Estimating D_{ff}'s Eigenvalues and Eigenvectors: In [20], a distributed algorithm is proposed to estimate x from a linear algebraic equation $Ax = b$, where A, b can be a pair of matrices. Based on [20], a distributed algorithm is proposed in [17] to compute both the eigenvalues and eigenvectors of an irreducible matrix associated with strongly connected digraphs. Inspired by [17], since $\bar{D}_{ff}$ is a nonsingular matrix for any triangular angularities, we first write

$$\bar{D}_{ff}\bar{D}_{ff}^{-1} = I_{2nf} \tag{14}$$

Note that $\bar{D}_{ff}$'s or D_{ff}'s $(2i-1)$th to $(2i)$th rows are known by free node i's angle measurements and communication with its neighbors. Also, the communication topology among the virtual nodes is described by $\bar{\mathcal{G}}_f$. Therefore, we can develop a distributed algorithm based on solving linear equations in [20] such that each virtual node can get the knowledge of $\bar{D}_{ff}^{-1}$. Here, we consider that each free node has the knowledge of n_f and γ, where the former can be achieved by using a distributed average consensus law [21, Sec. IV] and the latter can be preset.

Following [17, Sec. IV.B], we estimate $\bar{D}_{ff}^{-1}$ in each virtual node under the communication graph $\bar{\mathcal{G}}_f$. First, we assign each virtual node $i_{-j}, i \in \mathcal{V}_f, j \in \{1,2\}$ a state variable $Z_{i_{-j}} \in \mathbb{R}^{2n_f \times 2n_f}$, under which each free node i has two state variables $Z_{i_{-1}}$ and $Z_{i_{-2}}$. Then, each virtual node estimates $\bar{D}_{ff}^{-1}$ by executing the following updating rule:

$$Z_{i_{-j}}[k+1] = Z_{i_{-j}}[k] - \frac{1}{|\bar{\mathcal{N}}_{i_{-j}}|} P_{i_{-j}} \left(|\bar{\mathcal{N}}_{i_{-j}}| Z_{i_{-j}}[k] - \sum_{j \in \bar{\mathcal{N}}_{i_{-j}}} Z_j[k] \right) \tag{15}$$

where $\bar{\mathcal{N}}_{i_{-j}}$ represents the virtual node (i_{-j})'s neighbor set in graph $\bar{\mathcal{G}}_f$, and $P_{i_{-j}} = P_{i_{-j}}^\top \in \mathbb{R}^{2n_f \times 2n_f}$ is the orthogonal projection of the kernel of $[\bar{D}_{ff}]_{(2i-2+j)*}$, i.e.,

$$P_{i_{-j}} := I_{2n_f} - \frac{([\bar{D}_{ff}]_{(2i-2+j)*})^\top ([\bar{D}_{ff}]_{(2i-2+j)*})}{([\bar{D}_{ff}]_{(2i-2+j)*})([\bar{D}_{ff}]_{(2i-2+j)*})^\top} \tag{16}$$

Since $[\bar{D}_{ff}]_{2i*} = [D_{ff}]_{2i*} + \gamma e_{2i}^\top$ and γ is known, node i has the knowledge of $[\bar{D}_{ff}]_{2i*}$ and $[\bar{D}_{ff}]_{(2i-1)*}$. Thus, (15) is distributed since all the required information to execute (15) is either locally measured or obtained by communication with neighbors. Now, we present the following result.

Proposition 1: Let the initial condition $Z_{i_{-j}}[0]$ of (15) satisfy $([\bar{D}_{ff}]_{(2i-2+j)*})Z_{i_{-j}}[0] = [I_{2n_f}]_{(2i-2+j)*}$ for $\forall i \in \mathcal{V}_f, j \in \{1,2\}$. Then, under the distributed law (15) and Assumption 1, the state $Z_{i_{-j}}[k]$ exponentially converges to $\bar{D}_{ff}^{-1}$. Correspondingly, one has

$$\lambda_s(D_{ff}) = \lambda_s^{-1}(Z_{i_{-j}}^e) - \gamma, \quad \forall s = 1,\ldots,2n_f$$
$$v_s^l(D_{ff}) = v_s^l(Z_{i_{-j}}^e),\ v_s^r(D_{ff}) = v_s^r(Z_{i_{-j}}^e) \tag{17}$$

where $Z_{i_{-j}}^e$ represents the steady-state of $Z_{i_{-j}}[k]$. Moreover, the convergence of (15) is guaranteed if the graph in Assumption 1 is replaced by a repeatedly jointly strongly connected graph.

Under Assumption 1, the graph $\bar{\mathcal{G}}_f$ is connected. Since [20, Corollary 1] (respectively, [20, Th. 1]) holds for any strongly connected graphs (respectively, repeatedly jointly strongly connected graphs), the proof of Proposition 1 follows the same line as [20, Corollary 1, Th. 1]. According to [20, Corollary 1], there exists an estimate for the convergence speed of (15). Also, according to [20, Th. 4], the updating law (15) can be executed asynchronously. If n_f is a variable, then the dimension of the eigenvectors scales up with $\mathbb{O}(n_f)$. Note that node localizability is a property of the whole network, rather than a local property, and that is why the algorithm for checking node localizability requires to compute an eigenvector with its dimension scaling up with the number of free nodes.

Remark 2: The execution of updating law (15) is different from that of the estimation law in [17] since the communication topology $\bar{\mathcal{G}}_f$ for (15) is derived from $\mathcal{G}_f$ instead of D_{ff}'s nonzero elements. In addition to (15), some other existing distributed algorithms can also be used to solve linear equation (14) [22]. Although many distributed algorithms have been proposed to solve linear equation $Ax = b$ [22], the usage of them for checking node localizability has not been adequately studied, which is the contribution of this section.

Remark 3: Different from distance-based node localizability conditions [10], [11], [13], this work proposes an angle-based node localizability condition, which is necessary and sufficient. Compared with [10] and [13], this work proposes a distributed and angle-based node localizability checking algorithm. Compared with [11], which requires a sequential cluster-based execution, (15) is applicable for nonsequential and noncluster networks. Compared with [12], (15) has less communication cost. The developed checking condition in Theorem 1 is based on [12], while the difference between their checking algorithms is the usage of (17) in this article, which transforms the estimation of eigenvector $v_i^l(D_{ff})$ as the estimation of $v_i^l(\bar{D}_{ff}^{-1})$.

D. Improving Unlocalizable Networks' Localizability

Based on the distributed algorithm developed in the above section, each node knows whether it itself is localizable in the network by only communicating with its neighbors. In this section, we study how to add additional angle measurements for those unlocalizable nodes to make them also localizable. First, we present a lemma.

Lemma 5: For a triangular angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$, it is impossible that the angularity has only one unlocalizable node.

Proof: Suppose on the contrary that the network has only one unlocalizable node, which is denoted by i. There exists at least one triangle in $\mathcal{A}$ that is associated with i, which we denote by $\triangle ijm$, $\{j,m\} \subset \mathcal{V}$. Then, the angle-induced linear equation $f_i^{\triangle ijm}(\alpha, [p_i^\top, p_j^\top, p_m^\top]^\top) = 0$ guarantees that p_i is uniquely determined, which implies a contradiction to the assumption that i is unlocalizable. □

Theorem 2: If a triangular angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ has only two unlocalizable nodes i and j, then these two nodes must be simultaneously associated within only one triangle. Moreover, the triangular angularity $\mathbb{A}'(\mathcal{V},\mathcal{A}\cup\mathcal{A}_1,p)$ with $\mathcal{A}_1 = \{(i,j,m),(i,m,j),(j,i,m)\}$, $\mathcal{A}_1 \not\subseteq \mathcal{A}$, and $m \in \mathcal{V}$ must be localizable.

Proof: Suppose on the contrary that the nodes i and j are associated within $\mathcal{A}$'s two different triangles $\triangle i m_1 j_1$ and $\triangle j m_2 i_2$, respectively. According to Lemma 5, i and j must be localizable by applying the angle-induced linear equations in $\triangle i m_1 j_1$ and $\triangle j m_2 i_2$, which implies a contradiction.

Since $\mathcal{A}_1 \not\subseteq \mathcal{A}$, there exist $j_1 \in \mathcal{V}$, $m_1 \in \mathcal{V}$ with $j_1 \neq m$, $m_1 \neq m$ such that $(i, j_1, m_1) \in \mathcal{A}$. Since i and j are the only unlocalizable nodes, by using the angle-induced linear equations $f_i^{\triangle ijm}(\alpha,[p_i^\top,p_j^\top,p_m^\top]^\top) = 0$ and $f_i^{\triangle i j_1 m_1}(\alpha,[p_i^\top,p_{j_1}^\top,p_{m_1}^\top]^\top) = 0$, i and j are uniquely determined. Therefore, under the angle constraints in $\mathcal{A}_1 \cup \mathcal{A}$, $\mathbb{A}'$ is localizable. □

Indeed, three additional angle measurements associated within one triangle are needed to ensure the unlocalizable angularity with two unlocalizable nodes becoming localizable. However, when the network's topology is not properly designed, it might have many unlocalizable nodes. Therefore, we now discuss how to add angle measurements such that those networks with many unlocalizable nodes become localizable.

Suppose that the sensor network executes the node localizability checking algorithm in real time, under which each node knows whether it itself is localizable under the current network $\mathbb{A}(\mathcal{V},\mathcal{A}[k],p)$, where $\mathcal{A}[k]$ represents the network's triangular angle set at the iteration step $k \in \mathbb{N}$. Divide the vertex set $\mathcal{V} = \mathcal{V}_l \cup \mathcal{V}_u$ into a set $\mathcal{V}_l$ with localizable vertices and a set $\mathcal{V}_u$ with unlocalizable vertices, in which $\mathcal{V}_a \subset \mathcal{V}_l$. It holds that $\mathcal{V}_l[k] \cap \mathcal{V}_u[k] = \emptyset, |\mathcal{V}_l[k]| \geq 2, \forall k \geq 0$, and $|\mathcal{V}_u[0]| \geq 2$. The aim is to achieve $\forall k > \bar{k}, \mathcal{V}_u[k] = \emptyset$ by iteratively adding some triangular angle measurements into $\mathcal{A}[0]$, where $\bar{k}$ is a positive integer. We propose the following localizability improvement algorithm, which consists of two main steps.

Step 1 (Neighbor searching): If a node $i \in \mathcal{V}_u$ satisfies that $j \in \mathcal{N}_i$ and $j \in \mathcal{V}_l$, then i (or the other node forming a triangle with i and j) should search for a shortest path $\mathcal{P}_i = \{s_1, s_2, \ldots, s_q\}$ to another localizable node $m \neq j$; otherwise, i does not need to do anything. Such kind of i always exists in the network $\mathbb{A}(\mathcal{V},\mathcal{A}[k],p)$ with $\mathcal{V}_u[k] \neq \emptyset$ and $|\mathcal{V}_l[k]| \geq 2$. Also, those nodes in $\mathcal{P}_i$ satisfy that $s_1 \in \mathcal{N}_i, s_2 \in \mathcal{N}_{s_1}$,..., and $m \in \mathcal{N}_{s_q}$. Nodes $s_1, s_2, \ldots,$ and s_q can be seen as node i's 1-hop, 2-hop,..., and q-hop neighbors, respectively. We assume that if node i's q-hop neighbors are unlocalizable, then node i can get the information of its q-hop neighbor s_q's neighbors. Note that it is possible that $j \in \mathcal{P}_i$. The operational complexity for such kind of neighbor searching is $\mathbb{O}(n)$.

Step 2 (Angle addition): Node i needs to add some triangular angle measurements such that the induced subnetwork whose vertex set is $\mathcal{V}_{j-m} = \{j,i,m\} \cup \mathcal{P}_i$ becomes localizable. Let $\mathbb{A}_{j-m}(\mathcal{V}_{j-m},\mathcal{A}_{j-m},p')$ be the induced triangular angularity, where $\mathcal{A}_{j-m}$ is the triangular angle set associated with the vertices in $\mathcal{V}_{j-m}$, and $p' \in \mathbb{R}^{2|\mathcal{V}_{j-m}|}$ is the position vector of the vertices in $\mathcal{V}_{j-m}$. Consider that node i knows the information of $\mathcal{A}_{j-m}$ at Step 1. Obviously, j and m are the only localizable nodes in $\mathbb{A}_{j-m}$ (otherwise, there exists a path for node i to reach node m, which is shorter than $\mathcal{P}_i$). Denote by $\mathcal{A}_2$ the triangular angle set to be added such that $\mathbb{A}_2(\mathcal{V}_{j-m},\mathcal{A}_{j-m}\cup\mathcal{A}_2,p')$ is localizable. According to [7, Ths. 6 and 7], $\mathbb{A}_2$ is localizable if and only if $\mathbb{A}_2$ is triangularly angle rigid, which holds if and only if $(\mathcal{V}_{j-m},\mathcal{A}_{j-m}\cup\mathcal{A}_2)$ contains an L-trigraph $(\mathcal{V}_{j-m},\mathcal{A}_3)$ with $\mathcal{A}_3 \subseteq \mathcal{A}_{j-m}\cup\mathcal{A}_2$. Two conditions are needed to ensure that $(\mathcal{V}_{j-m},\mathcal{A}_{j-m}\cup\mathcal{A}_2)$ contains an L-trigraph. The first condition is to ensure $n^{\triangle}_{\mathcal{A}_{j-m}\cup\mathcal{A}_2} \geq |\mathcal{V}_{j-m}| - 2$, and the second condition is to ensure that for every subtrigraph $(\mathcal{V}_4,\mathcal{A}_4)$ of $(\mathcal{V}_{j-m},\mathcal{A}_3)$, $n^{\triangle}_{\mathcal{A}_4} \leq |\mathcal{V}_4| - 2$. Note that the set $\mathcal{V}_{j-m}$ only consists

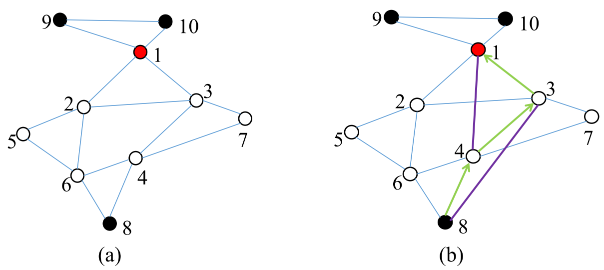

Fig. 2. Improving unlocalizable networks' localizability. (a) Network with three anchors (black), one localizable nodes (red), and six unlocalizable nodes (white), $\bar{\mathcal{A}} = \{(1,9,10),(1,2,3),(2,5,6),(4,7,3),(6,8,4)\}$. (b) At Step 1, node 6 finds a path 4→ 3 →1, and adds two triangles $\{(3,4,8),(1,3,4)\} \in \mathcal{A}_2$. After the addition, all nodes are localizable.

of those nodes in the path. To make the addition of triangular angle measurements more efficient, in node i's neighbor-searching stage (Step 1), if $\exists \mathbb{A}_5(\mathcal{V}_5, \mathcal{A}_5, p'')$ with $\mathcal{V}_5 \cap \mathcal{V}_{j-m} \neq \emptyset, \mathcal{A}_5 \cap \mathcal{A}_{j-m} = \emptyset$ and $(\mathcal{V}_5, \mathcal{A}_5)$ itself being an L-trigraph, then the above induced triangular angularity $\mathbb{A}_{j-m}(\mathcal{V}_{j-m}, \mathcal{A}_{j-m}, p')$ should be replaced by $\mathbb{A}'_{j-m}(\mathcal{V}_{j-m} \cup \mathcal{V}_5, \mathcal{A}_{j-m} \cup \mathcal{A}_5, [p'^{\top}, p''^{\top}]^{\top})$.

An example is given in Fig. 2 to explain the above two steps. By iteratively executing the two steps, there exists a $\bar{k} \in \mathbb{N}^+$ such that $\forall k > \bar{k}, \mathcal{V}_u[k] = \emptyset$. If each neighbor-searching operation in Step 1 is counted as one iteration, an upper bound of $\bar{k}$ is $\mathcal{V}_u[0]$. This is because if the shortest path $\mathcal{P}_i$ is found with k iterations, then after the addition of triangular angle measurements, at least k unlocalizable nodes in $\mathcal{A}_{j-m}$ become localizable. Another factor that needs to be considered is to avoid some repetitive neighbor-searching operations, e.g., in Fig. 2, although nodes 2–4 and 6 can all execute neighbor-searching operations, a neighbor-searching operation from only one of them is sufficient.

Remark 4: It is worth emphasizing that the networks we aim to study in this article are *triangular* networks [7] instead of *triangulated Laman* networks which are constructed in a sequential manner and started from a single edge with two nodes. Hence, triangulated Laman networks are always localizable by selecting the first two connected nodes as anchor nodes, which is not the case for triangular networks; see Fig. 2(a) as an example. Indeed, many types of sensor networks cannot be triangulated and the anchor nodes may not be connected. This motivated the study of the localizability of triangular networks.

IV. Distributed Localization

As mentioned in Remark 1, the eigenvalue information $\lambda_{\max}(D_{ff})$ required for the selection of the sampling period h in (6) is global and graphic information. Instead of employing communication resources (e.g., by executing maximum consensus algorithms in [6, Algorithm 2]) to obtain an upper bound of $\lambda_{\max}(D_{ff})$, we aim to propose a fully distributed localization law by using the information of each sensor node's triangle degree in the network, which is locally known by each node. Specifically, by replacing h with $1/2d_i$, we modify (5) into

$$\hat{p}_i[k+1] = \hat{p}_i[k] - \frac{1}{2d_i}\left(F_i^{\triangle ij_1m_1} + F_i^{\triangle j_2im_2} + F_i^{\triangle j_3m_3i}\right) \quad (18)$$

where $d_i \in \mathbb{N}^+$ is the triangle degree of node i in the triangular angularity $\mathbb{A}$, and $F_i^{\triangle ij_1m_1}, F_i^{\triangle j_2im_2}$, and $F_i^{\triangle j_3m_3i}$ follow the same definitions as those in (5).

Now, we present the main result on the convergence of (18).

Theorem 3: If the triangular angularity $\mathcal{A}(\mathcal{V}, \mathcal{A}, p)$ is localizable, then $\hat{p}_f[k]$ globally and exponentially converges to p_f under the fully distributed localization algorithm (18).

Proof: Writing (18) into a compact form yields

$$\tilde{p}_f[k+1] = \left(I_{2n_f} - (0.5\text{diag}\{d_i^{-1}\} \otimes I_2)D_{ff}\right)\tilde{p}_f[k] \quad (19)$$

where $\tilde{p}_f[k] = \hat{p}_f[k] - p_f$ and $\text{diag}\{d_i^{-1}\} = \text{diag}\{[d_1^{-1}, \ldots, d_{n_f}^{-1}]\} \in \mathbb{R}^{n_f \times n_f}$. The stability of (19) depends on the distribution of the eigenvalues of $(0.5\text{diag}\{d_i^{-1}\} \otimes I_2)D_{ff}$, which should lie in (0,2) if they are real. Since $0.5\text{diag}\{d_i^{-1}\}$ and D_{ff} are both positive definite, according to [23, Th. 10.31], one has that all the eigenvalues of $(0.5\text{diag}\{d_i^{-1}\} \otimes I_2)D_{ff}$ are real and positive. Therefore, it follows that

$$\lambda_{\min}\left(0.5\text{diag}\{d_i^{-1}\} \otimes I_2 D_{ff}\right) > 0. \quad (20)$$

Then, we discuss the magnitude of the maximum eigenvalue of $(0.5\text{diag}\{d_i^{-1}\} \otimes I_2)D_{ff}$. According to the Gershgorin circle theorem [24, Th. 6.1.1] and the structural properties in Lemma 3, all the eigenvalues λ of $(0.5\text{diag}\{d_i^{-1}\} \otimes I_2)D_{ff}$ must lie within the union of the following $2n_f$ disks:

$$\left|\lambda - \frac{\sum_{j_1,m_1}\beta_1^{\triangle i_sj_sm_s}}{2d_i}\right| \leq \frac{\sum_{j_1,m_1}(|\beta_2^{\triangle i_sj_sm_s}| + |\beta_3^{\triangle i_sj_sm_s}|)}{2d_i} \quad (21)$$

where $i \in \mathcal{V}_f$, $\{i, j_1, m_1\} = \{i_s, j_s, m_s\}$, $(i_s, j_s, m_s) \in \bar{\mathcal{A}}$, $\frac{\sum_{j_1,m_1}\beta_1^{\triangle i_sj_sm_s}}{2d_i} \in \mathbb{R}$ corresponds to the ith diagonal element, and $\frac{\sum_{j_1,m_1}(|\beta_2^{\triangle i_sj_sm_s}|+|\beta_3^{\triangle i_sj_sm_s}|)}{2d_i} \in \mathbb{R}$ corresponds to the sum of those off-diagonal elements at the $(2i-1)$th or $(2i)$th row of the matrix $(0.5\text{diag}\{d_i^{-1}\} \otimes I_2)D_{ff}$. Using the structural properties in Lemma 3, one has

$$0 < \frac{\sum_{j_1,m_1}\beta_1^{\triangle i_sj_sm_s}}{2d_i} < \frac{d_i}{2d_i} = 0.5$$
$$0 < \frac{\sum_{j_1,m_1}(|\beta_2^{\triangle i_sj_sm_s}| + |\beta_3^{\triangle i_sj_sm_s}|)}{2d_i} < \frac{2\sqrt{2}d_i}{2d_i} = \sqrt{2} \quad (22)$$

where the usage of $2\sqrt{2}d_i$ is because if $j_1 \in \mathcal{V}_f$ and $m_1 \in \mathcal{V}_f$, then D_{ff}'s off-diagonal blocks of the $(2i-1)$th to $(2i)$th rows include both $(A_i^{\triangle ij_1m_1})^{\top} A_{j_1}^{\triangle ij_1m_1}$ and $(A_i^{\triangle ij_1m_1})^{\top} A_{m_1}^{\triangle ij_1m_1}$, which can be seen from (8). Substituting (22) into (21) yields

$$\lambda_{\max}\left(0.5\text{diag}\{d_i^{-1}\} \otimes I_2 D_{ff}\right) < 0.5 + \sqrt{2} < 2. \quad (23)$$

Combining (20) and (23), one has that all the eigenvalues of $I_{2n_f} - (0.5\text{diag}\{d_i^{-1}\} \otimes I_2)D_{ff}$ lie within the unit disk, which implies the global and exponential convergence of (19). □

Remark 5: Compared with continuous localization [8], [14], discrete localization algorithms have less communication burden. Compared with bearing-based localization [5], [15], [25], the angle-based localization algorithm (18) does not need to estimate/know the orientations of the sensor nodes' coordinate frames. Compared with our discrete localization law in [7], (18) is fully distributed since it does not require the global information (6) related to h that needs to satisfy.

V. Simulations

In this section, we conduct simulations on a network with 20 sensor nodes to demonstrate the node localizability checking algorithm, the localizability improvement algorithm, and the fully distributed localization algorithm.

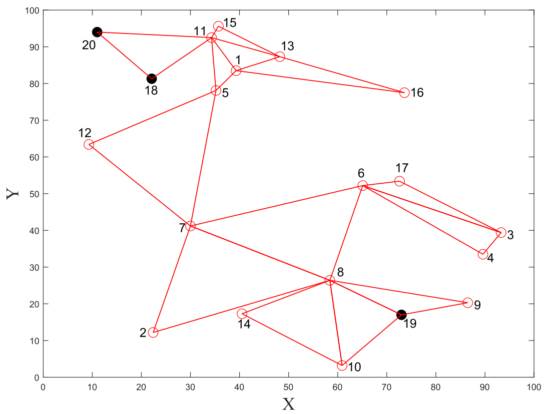

Fig. 3. Network $\mathbb{A}$ with 17 free nodes and three anchor nodes.

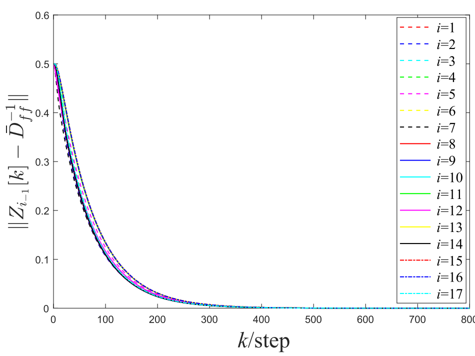

Fig. 4. Evolution of error $\|Z_{i_{-1}}[k] - \bar{D}_{ff}^{-1}\|, i = 1, \ldots, 17$.

A. Node Localizability Checking Algorithm

As shown in Fig. 3, we randomly produce a sensor network $\mathbb{A}$ with 17 free nodes labeled from 1 to 17, three anchor nodes labeled from 18 to 20, and 12 triangles $\bar{\mathcal{A}} = (20, 18, 11), (11, 15, 13), (1, 5, 11), (1, 13, 16), (5, 12, 7), (2, 7, 8), (6, 7, 8), (3, 4, 6), (3, 6, 17), (8, 10, 14), (8, 10, 19), (8, 19, 9)$. It is obvious that only node 11 is localizable.

First, we employ Theorem 1 to check the node localizability. By using the information of those angles defined in $\bar{\mathcal{A}}$, one has $\lambda_i(D_{ff}) = 0\, \forall i = 1, \ldots, 10$, and $\lambda_j(D_{ff}) > 0\, \forall j = 11, \ldots, 34$. Also, by numerical computations, the kernel of D_{ff} consists of ten eigenvectors. The 21st row and 22nd row of these ten eigenvectors are $10^{-16} * [8, 2, 6, -6, -1, -1, 6, 2, -9, -6]$ and $10^{-16} * [2, 6, 1, -4, -2, 9, 2, -9, 8, -1]$, respectively. Since there are no other zero rows in these ten eigenvectors, this kernel information validates that node 11 is localizable, and the other free nodes are unlocalizable.

Then, we execute the algorithm (15) to check the node localizability in a distributed manner. Fig. 4 shows the evolution of error $\|Z_{i_{-1}}[k] - \bar{D}_{ff}^{-1}\|, i = 1, \ldots, 17$. From Fig. 4, the convergence is reached after around 300 iteration steps. After this convergence, each node has the knowledge of $\bar{D}_{ff}^{-1}$, from which each free node has the knowledge of all the eigenvalues and eigenvectors of D_{ff} by (17) and, thus, can determine its node localizability in the network $\mathbb{A}$.

B. Localizability Improvement Algorithm

We add triangular angle measurements into network $\mathbb{A}$ such that all the free nodes become localizable. By using the localizability

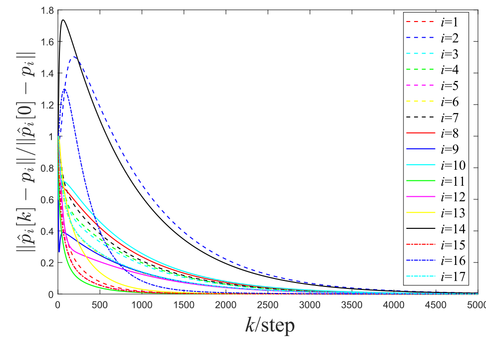

Fig. 5. Position estimation error under (18).

improvement algorithm in Section III-D, the following iteration steps can be executed.

Step 1: Node 5 finds a path $5 \rightarrow 11 \rightarrow 18$, and adds triangle (5,11,18), after which nodes 1 and 5 become localizable.

Step 2: Node 13 finds a path $13 \rightarrow 11 \rightarrow 1$, and adds triangle (1,11,13), after which nodes 13, 15, and 16 become localizable.

Step 3: Node 8 finds a path $8 \rightarrow 7 \rightarrow 5$. To efficiently conduct the angle-addition operation, we consider a larger network $\mathbb{A}'_{19-5}$ with vertices $\{19, 9, 10, 14, 8, 7, 6, 2, 5, 12\}$ and triangles (5,12,7), (2,7,8), (6,7,8), (8,10,14), (8,19,9), (8,10,19). Obviously, there are only one localizable node and one anchor node in $\mathbb{A}_{19-5}$. Thus, $\mathbb{A}'_{19-5}$ is localizable if and only if it is triangularly angle rigid, which holds if and only if $\mathbb{A}'_{19-5}$ spans an L-trigraph. Based on this, one can add triangles (5,6,7), (2,8,14) into $\mathbb{A}'_{19-5}$, after which all the vertices in $\mathbb{A}'_{19-5}$ become localizable.

Step 4: Node 4 finds a path $4 \rightarrow 6 \rightarrow 8$, and adds triangle (4,6,8), after which nodes 3, 4, and 17 become localizable.

To sum up, by adding five triangles into $\mathbb{A}$ in the above four steps, all the free nodes and the entire network become localizable, which is verified by using Lemma 1.

C. Distributed Localization Algorithm

After all the nodes are localizable, we execute the localization algorithm (18) for the obtained new network in Section V-B. The localization error is shown in Fig. 5, which converges to zero after around 4500 iteration steps.

VI. Conclusion

This article proposed angle-based distributed algorithms to check node localizability and achieve localization. First, an algebraic condition was proposed to check node localizability, based on which a distributed node localizability checking algorithm was proposed. Then, to make those unlocalizable nodes localizable, we proposed a strategy to add triangular angle measurements for those unlocalizable nodes. Finally, a fully distributed localization algorithm was proposed, which does not rely on any globally graphic information. As future work, it is interesting to study distributed node localizability for other types of sensor networks, such as bearing-based and distance-based sensor networks.

References

[1] T.-M. Nguyen, Z. Qiu, T. H. Nguyen, M. Cao, and L. Xie, "Persistently excited adaptive relative localization and time-varying formation of robot swarms," *IEEE Trans. Robot.*, vol. 36, no. 2, pp. 553–560, Apr. 2020.

[2] G. Mao, B. Fidan, and B. D. Anderson, "Wireless sensor network localization techniques," *Comput. Netw.*, vol. 51, no. 10, pp. 2529–2553, 2007.

[3] Z. Lin, M. Fu, and Y. Diao, "Distributed self localization for relative position sensing networks in 2D space," *IEEE Trans. Signal Process.*, vol. 63, no. 14, pp. 3751–3761, Jul. 2015.

[4] Y. Diao, Z. Lin, and M. Fu, "A barycentric coordinate based distributed localization algorithm for sensor networks," *IEEE Trans. Signal Process.*, vol. 62, no. 18, pp. 4760–4771, Sep. 2014.

[5] S. Zhao and D. Zelazo, "Localizability and distributed protocols for bearing-based network localization in arbitrary dimensions," *Automatica*, vol. 69, pp. 334–341, 2016.

[6] Z. Lin, T. Han, R. Zheng, and M. Fu, "Distributed localization for 2-D sensor networks with bearing-only measurements under switching topologies," *IEEE Trans. Signal Process.*, vol. 64, no. 23, pp. 6345–6359, Dec. 2016.

[7] L. Chen, "Triangular angle rigidity for distributed localization in 2D," *Automatica*, vol. 143, 2022, Art. no. 110414.

[8] G. Jing, C. Wan, and R. Dai, "Angle-based sensor network localization," *IEEE Trans. Autom. Control*, vol. 67, no. 2, pp. 840–855, Feb. 2022.

[9] L. Chen, K. Cao, L. Xie, X. Li, and M. Feroskhan, "3D network localization using angle measurements and reduced communication," *IEEE Trans. Signal Process.*, vol. 70, pp. 2402–2415, 2022.

[10] Z. Yang and Y. Liu, "Understanding node localizability of wireless ad hoc and sensor networks," *IEEE Trans. Mobile Comput.*, vol. 11, no. 8, pp. 1249–1260, Aug. 2012.

[11] Y. Diao, M. Fu, Z. Lin, and H. Zhang, "A sequential cluster-based approach to node localizability of sensor networks," *IEEE Trans. Control Netw. Syst.*, vol. 2, no. 4, pp. 358–369, Dec. 2015.

[12] Z. Lin, Z. Han, and M. Cao, "Distributed localization for multi-robot systems in presence of unlocalizable robots," in *Proc. IEEE 59th Conf. Decis. Control*, 2020, pp. 1550–1555.

[13] H. Ping, Y. Wang, X. Shen, D. Li, and W. Chen, "On node localizability identification in barycentric linear localization," *ACM Trans. Sensor Netw.*, vol. 19, pp. 1–26, 2022.

[14] X. Fang, X. Li, and L. Xie, "Angle-displacement rigidity theory with application to distributed network localization," *IEEE Trans. Autom. Control*, vol. 66, no. 6, pp. 2574–2587, Jun. 2021.

[15] X. Li, X. Luo, and S. Zhao, "Globally convergent distributed network localization using locally measured bearings," *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 1, pp. 245–253, Mar. 2020.

[16] M. A. Rahimian, A. Ajorlou, R. Tutunov, and A. G. Aghdam, "Identifiability of links and nodes in multi-agent systems under the agreement protocol," in *Proc. Amer. Control Conf.*, 2013, pp. 6853–6858.

[17] A. Gusrialdi and Z. Qu, "Distributed estimation of all the eigenvalues and eigenvectors of matrices associated with strongly connected digraphs," *IEEE Control Syst. Lett.*, vol. 1, no. 2, pp. 328–333, Oct. 2017.

[18] T. Charalambous, M. G. Rabbat, M. Johansson, and C. N. Hadjicostis, "Distributed finite-time computation of digraph parameters: Left-eigenvector, out-degree and spectrum," *IEEE Trans. Control Netw. Syst.*, vol. 3, no. 2, pp. 137–148, Jun. 2016.

[19] M. Franceschelli, A. Gasparri, A. Giua, and C. Seatzu, "Decentralized estimation of Laplacian eigenvalues in multi-agent systems," *Automatica*, vol. 49, no. 4, pp. 1031–1036, 2013.

[20] S. Mou, J. Liu, and A. S. Morse, "A distributed algorithm for solving a linear algebraic equation," *IEEE Trans. Autom. Control*, vol. 60, no. 11, pp. 2863–2878, Nov. 2015.

[21] S. Zhang, C. Tepedelenlioğlu, M. K. Banavar, and A. Spanias, "Distributed node counting in wireless sensor networks in the presence of communication noise," *IEEE Sens. J.*, vol. 17, no. 4, pp. 1175–1186, Feb. 2017.

[22] P. Wang, S. Mou, J. Lian, and W. Ren, "Solving a system of linear equations: From centralized to distributed algorithms," *Annu. Rev. Control*, vol. 47, pp. 306–322, 2019.

[23] F. Zhang, *Matrix Theory: Basic Results and Techniques*. New York, NY, USA: Springer, 2011.

[24] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 2012.

[25] A. N. Bishop, B. D. Anderson, B. Fidan, P. N. Pathirana, and G. Mao, "Bearing-only localization using geometrically constrained optimization," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 45, no. 1, pp. 308–320, Jan. 2009.