Startup fundraising and equity split under double-sided moral hazard with a two-stage investment

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Abstract

This paper examines optimal equity split between an entrepreneur (E) without capital and a venture capitalist (V) under double-sided moral hazard with a two-stage investment in a project. The first-stage investment explores project profitability and the final output is a Cobb-Douglas production function of V’s investment scale, E’s and V’s private effort. We show that if profit prospects revealed or on average are good enough, optimal equity split and the welfare loss rate arising from moral hazard are explicitly determined only by inputs’ output elasticities. Otherwise, V’s share would decrease in project profitability. If the contract can be renegotiable, after the profitability is revealed, we specify the thresholds determining whether E should abandon the project, whether E should go ahead without doing anything, and whether E should increase V’s equity or transfer cash to V until V’s participation constraint is satisfied.

Keywords: Optimal contracting; Double-sided moral hazard; Venture capital; Equity split; Renegotiation.

JEL: D82, G24, M13

1. Introduction

Startups and entrepreneurial firms serve as important engines for economic growth. According to Cumming et al. (2022), in the United States, they represent 99.9% of all firms. Many of them choose to invest in a high risk project for acquiring a disruptive

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or leading technique. Mainly due to information asymmetry and high risk, it is difficult for startups to obtain financing in traditional instruments. Venture capital is an important source of external funds for startups and innovative entrepreneurship. Financing in venture capital markets inevitably involves share allocation (equity split) between an entrepreneur (E) who has innovative ideas but no capital and a venture capitalist (V) who has unlimited capital but no ideas.\(^1\)

To reduce investment risk, multi-stage investments are often conducted. Without much loss of generality, we focus on a two-stage investment. The first-stage investment aims to explore project profitability by paying a small amount of sunk costs. After the project profitability is revealed, the final project output is determined by the investment scale provided by V as well as both E’s and V’s private effort. A challenging problem arises: How to split equity between E and V. This is related to optimal financial contracting under double-sided moral hazard. We must provide sufficient incentives for both E’s and V’s effort as well as V’s two-stage investment.

Studies about venture capital markets have recorded considerable growth in recent decades with a focus on corporate securities’ roles in designing venture capital contracts, neglecting how capital and labour interact with each other to make project success. Clearly, capital and participants’ efforts are two indispensable determinants to make a project success. No capital or no effort, no project output. The importance of capital for ventures’ success has been stressed in the literature, see, e.g., Edelman et al. (2005) and Powers and McDougall (2005)). The well-known Cobb-Douglas production function states clearly the indispensable role of capital as well as labour (effort) for project output. The more the investment, the more the project output, or equivalently the higher the probability of venture capital success. Motivated by these observations, we borrow ideas from the Cobb-Douglas production function to measure project success’s probability and take as inputs both E’s and V’s effort as well as the capital provided by V.

In our model, V plays dual roles. V provides funds as well as actively participation in the management of the startup he funds, see, e.g. Sahlman (1990): V uses his

\(^1\)Hereinafter, the entrepreneur may be referred to as the pronoun “she” or “E” while “he” or “V” refers to the venture capitalist.
experience, contacts and reputation to provide advice to E in terms of selecting qualified personnel or dealing with suppliers and customers. E with core technology, e.g., a scientist, who is wealth constrained and protected by limited liability, should also make efforts to turn technology into project output. E’s private effort and V’s private advice and investment are private information. As a result, information asymmetry between the two participants arises, leading to a double-sided hazard risk problem.

We address optimal shares (equity split) of final returns distributed to participants in both single-stage financing and two-stage financing with interim information. To design an optimal contract, we use a backward induction method. We begin by examining equity split for the second-stage financing. We then proceed to solve for two types of optimal contracts in the two-stage financing setting: A non-negotiable contract where the two parties sign a long-term agreement specifying the terms of the initial investment and outlining the following investment, contingent upon the profitability information, and a negotiable contract after project profitability disclosure, due to the incompleteness of ex ante contract, leading to a Pareto improvement if the renegotiation is successful. To be specific, we assume E possesses all bargaining power and receives all the renegotiation surplus except a necessary value transfer to meet V’s participant constraint. We then use a backward recursion method to design optimal contract under different situations.


The renegotiation we consider is related to incomplete contract theory originated

Our paper is most closely related to Repullo and Suarez (2004). Their paper also addresses venture capital finance in a double-sided moral hazard setting, but it is subject to a constant investment scale. The major differences between their paper and ours are as follows. First, instead of their constant investment cost, we assume that the investment scale is variable and endogenously decided by V. Along with the stakeholders’ effort level, the investment scale is an important determinant of project output or equivalently, the probability of project success. This assumption is realistic in practice as predicted by the well-known Cobb-Douglas production function. Second, to deal with incomplete contract case, we develop a renegotiation model in designing optimal contract. We show that the renegotiation is necessary only when V’s participation constraint is not satisfied after project profitability is revealed. Explicit renegotiation results are derived. Third, taking investment scale as a decision variable, we get the optimal contract in which the equity split is independent of project profitability and explicitly determined by inputs’ output elasticities. Last, we produce an explicit formula to measure the social welfare loss rate which is totally determined by the inputs’ output elasticities. These differences also fundamentally reflect our contributions to the literature on venture capital and financial contracting theory.

The remaining of the paper is organized as follows. Section 2 sets up the model. Section 3 focuses on optimal equity split between E and V after project profitability
is revealed. Section 4 explores the equity split before the project profitability is revealed, supposing that the contract is non-negotiable. Section 5 considers the equity split if the contract is negotiable. Section 6 concludes.

2. Model setup

We consider a game model under double-sided morel hazard with two-stage investment in a project. There are two players: an entrepreneur (E) endowed with an innovative project but no capital and a venture capitalist (V) who jointly plays a dual role as financier as well as investment advisor. E owns intellectual property such as business ideas and product patents, providing unobservable effort \( e \) in addition. V has accumulated business experience and business network, providing unobservable effort \( a \). V has sufficient capital to invest in the project.

To reduce investment risk, the project involves a two-stage investment. The first-stage investment requires a deterministic sunk cost \( k_0 \) paid by V to explore the project’s profitability, which is described by a random variable \( \theta \), a possible lump-sum payoff at the terminal time after the second-stage investment. If the profitability revealed after the first-stage investment is not good enough, the project is terminated with zero return. Otherwise, the second-stage investment is exercised after V has paid another sunk cost \( k \) decided by V. The probability of successfully realizing the profitability \( \theta \) depends on E’s effort \( e \), V’s effort \( a \) and sunk cost \( k \). The levels of effort are players’ private information, and all players get nothing if the second-stage investment fails.

We assume that the random potential profitability \( \theta \) takes values in the interval \( \Theta = [\theta_L, \theta_H] \) with \( \theta_L > 0 \). The terminal stochastic income \( X \) is described by the following two-point probability distribution:

\[
X = \begin{cases} 
0, & 1 - p(e, a, k), \\
\theta, & p(e, a, k),
\end{cases}
\]

\[(1)\]

\[\text{The participants’ actions change project profit generally in two ways: One changes the possible values of the profit and the other changes its probability distribution. We take the second way as in Holmstrom and Milgrom (1987) and in Repullo and Suarez (2004).}\]
with the success probability \( p(e, a, k) \) given by

\[
p(e, a, k) = Ak^{\alpha_k}a^{\alpha_a}e^{(1-\alpha_k-\alpha_a)},
\]

where \( k \in K = [0, \infty) \) refers to the second-stage investment cost provided by V (capital), \( a \in A = [0, \infty) \) indicates V’s advice level (labour), \( e \in E \equiv [0, \infty) \) represents E’s effort level (labour), \( \alpha_k, \alpha_a \in (0, 1) \), \( 1 - (\alpha_k + \alpha_a) \in (0, 1) \) characterize their output elasticities respectively, and \( A > 0 \) reflects the exogenous technique level.

We emphasize that in contrast to Repullo and Suarez (2004), the successful probability depends on the second-stage investment cost \( k \) which is totally paid and freely decided by V. This is a novelty of our model. These assumptions are quite in agreement with the Cobb-Douglas production function model that is the most widely used production function because it allows different combination of labour and capital. If either human capital (captured by effort \( e \) and \( a \)) or V’s capital (\( k \)) is missing, no output is produced and no value creation is possible. Our model assumption is also consistent with that assumed in Amit et al. (1990) and in Edmans and Gabaix (2016). By contrast, Repullo and Suarez (2004) assume that the capital \( k \) is constant, i.e. the capital factor is actually not considered.

Following Repullo and Suarez (2004), we assume that E’s and V’s effort cost functions are measured in the same unit as their payoff, given by

\[
U(e) = \frac{e^2}{2u}, \quad V(a) = \frac{a^2}{2v}
\]

respectively, where cost coefficients \( u > 0 \) and \( v > 0 \) are constant.

Once the contract is terminated, V receives payoff \( R \geq 0 \) from an external option (i.e. V’s opportunity cost of his entering into the contract). Therefore, V must get at least net profit \( R \) from the game and otherwise, he will not participate. E also has an outside option but its value is normalized to be zero on account of the fact that E is wealth-constrained and protected by the limited liability.

Fig. 1 displays the timeline of the model that provides a very fast graphical overview of the investment process. The aim of the first-stage (initial) investment is to explore how much the profitability is and the one of the second-stage investment
is to realize the potential profit if it is exercised. Next, we address how to allocate the profit between E and V.

<table>
<thead>
<tr>
<th>Initial investment</th>
<th>Information revealed</th>
<th>Effort of E and V</th>
<th>Return realized</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_0$</td>
<td>$\theta$</td>
<td>$(e, a)$</td>
<td>$X$</td>
</tr>
</tbody>
</table>

$k$

second-stage investment

**Figure 1. The timeline of the model**

3. **Optimal contract after the project profitability is revealed**

Our model involves a double-sided moral hazard problem. V provides both investment advice and capital, and thus he has two decision variables while E has only one. Using the backward induction method, we start from investigating the optimal contract at the second (expansion) stage, when the potential profitability $\theta \in [\theta_L, \theta_H]$ is revealed and publicly known.

3.1. **Benchmark model: optimal contract with full information**

In the full information case, all decisions are specified by a 'social planner' to maximize total social welfare. As a benchmark model, we derive the optimal effort levels exerted by E and V and optimal investment cost paid by V during the second stage, pretending that there is a social planner to determine all the decisions for maximizing the project value. Therefore, we need to solve the following optimization problem:

$$\max_{e, a, k} \left\{ Ak^{\alpha_k} a^{\alpha_a} e^{(1-\alpha_k-\alpha_a)} \theta - \frac{e^2}{2u} - \frac{a^2}{2v} - k \right\}. \tag{4}$$

Thanks to the first-order conditions (FOCs), we get

$$e^{FB}(\theta) = (\alpha_a v)^{\frac{\alpha_a}{2(1-\alpha_k)}} \left[u(1 - \alpha_k - \alpha_a)\right]^{\frac{2-\alpha_a-2\alpha_k}{2(1-\alpha_k)}} \alpha_k^{\frac{\alpha_k}{1-\alpha_k}} (A\theta)^{\frac{1}{1-\alpha_k}},$$

$$a^{FB}(\theta) = (\alpha_a v)^{\frac{1-\alpha_k+\alpha_a}{2(1-\alpha_k)}} \left[u(1 - \alpha_k - \alpha_a)\right]^{\frac{1-\alpha_a-\alpha_k}{2(1-\alpha_k)}} \alpha_k^{\frac{\alpha_k}{1-\alpha_k}} (A\theta)^{\frac{1}{1-\alpha_k}},$$
and

\[ k^{FB}(\theta) = (\alpha_a v)^{\frac{\alpha a}{1 - \alpha k}} [u(1 - \alpha_k - \alpha_a)]^{\frac{1 - \alpha_a - \alpha_k}{(1 - \alpha_k) \alpha_k^{1 - \alpha_a}}} (A\theta)^{1 - \alpha k}. \]

As a consequence, the first-best project value is then given by

\[ \Pi^{FB}(\theta) = (\alpha_a v)^{\frac{\alpha a}{1 - \alpha k}} [u(1 - \alpha_k - \alpha_a)]^{\frac{1 - \alpha_a - \alpha_k}{(1 - \alpha_k) \alpha_k^{1 - \alpha_a}}} (A\theta)^{1 - \alpha k} \left( \frac{1}{2} - \frac{1}{2} \alpha_k \right). \quad (5) \]

Clearly, we have \( \Pi^{FB}(\theta) > 0 \), and so the project is always profitable under the first-best strategies.

However, the above globally optimal strategy does not take players’ participant constraints into account. Therefore, we consider the contract \((s(\theta), 1 - s(\theta))\) that specifies the share \(0 < s(\theta) < 1\) of equity allocated to V while the remaining is allocated to E. The contract is signed at the very beginning of the second-stage investment. V’s participation constraint at the second stage is

\[ \Pi_V(s(\theta), \theta) = A k^{\alpha_k} a^{\alpha_a} e^{(1 - \alpha_k - \alpha_a)} \theta s(\theta) - \frac{a^2}{2v} - k \geq R, \quad (6) \]

and E’s participation constraint is

\[ \Pi_E(s(\theta), \theta) = A k^{\alpha_k} a^{\alpha_a} e^{(1 - \alpha_k - \alpha_a)} \theta (1 - s(\theta)) - \frac{e^2}{2v} \geq 0. \quad (7) \]

Naturally, only if the project value is greater than V’s outside option value, i.e. \( \Pi^{FB}(\theta) \geq R \), the project is feasible in the second-stage investment. We have the following proposition.

**Proposition 3.1.** The project is feasible at the second stage if and only if the profitability \( \theta \) revealed satisfies:

\[ \theta \geq \frac{1}{A} (\alpha_a v)^{-\frac{\alpha_a}{2}} [u(1 - \alpha_k - \alpha_a)]^{-\frac{1 - \alpha_a - \alpha_k}{2(1 - \alpha_k) \alpha_k^{1 - \alpha_a}}} A^{\frac{\alpha_k}{1 - \alpha_k}} 2R^{\frac{1 - \alpha_k}{1 - \alpha_k}}. \]

Suppose the project is feasible and then the share \( s(\theta) \) of equity allocated to V satisfies both E’s and V’s participant constraint if and only if

\[ s(\theta) \leq s(\theta) \leq \frac{1}{2} (1 + \alpha_a + \alpha_k), \]

where \( \frac{1}{2} (1 + \alpha_a + \alpha_k) \) is the maximum equity allocated to V to satisfy E’s participant constraint, and \( s(\theta) \) is the minimum share of equity allocated to V to meet his
participant constraint. We have

\[ 0 < s(\theta) = R(\alpha_a v)^{-\frac{\alpha_a}{1-\alpha_k}} [u(1 - \alpha_k - \alpha_a)^{-\frac{1}{1-\alpha_k}} - \alpha_k^{-\frac{2}{1-\alpha_k}} (A\theta)^{-\frac{2}{1-\alpha_k}} + \frac{\alpha_a}{2} + \alpha_k < 1. \]

To execute the second-stage investment, the project must be feasible. Generally, there are infinitely many contracts satisfying players’ participant constraints. How to determine the specific one depends on players’ bargaining power. We address this problem in a more realistic moral hazard model in the next subsection.

3.2. Optimal contract with hidden actions

Actually, there is no social planner in practice to make the decisions. The efforts exerted by players are decision-makers’ private information, and they are neither observable nor contractible. In addition, we suppose that the investment cost paid by V at the second stage is freely decided by V. Therefore, it is necessary to design an incentive-compatible contract against the players’ hidden actions that maximizes the project value.

Without loss of generality, we assume the venture capital market is fully competitive, and so V’s payoff is just \( R \) in equilibrium as before. Accordingly, it would be better for V to transfer some wealth \( T(\theta) \) to E in addition to E’s share \( 1 - s(\theta) \) of equity allocated.\(^3\) Therefore, a candidate contract at the second stage is defined by pair \( \{ s(\theta), T(\theta) \} \), where \( s(\theta) \in (0, 1) \) represents V’s share of equity allocated. E’s expected payoff from the contract after the second-stage investment is

\[ V_E(e; \theta) = Ak^{\alpha_k} a^{\alpha_a} e^{(1-\alpha_k-\alpha_a)} \theta (1 - s(\theta)) - \frac{e^2}{2u} + T(\theta), \quad (8) \]

while V’s expected payoff is

\[ V_V(a, k; \theta) = Ak^{\alpha_k} a^{\alpha_a} e^{(1-\alpha_k-\alpha_a)} \theta s(\theta) - \frac{\alpha^2}{2u} - k - T(\theta). \quad (9) \]

\(^3\) This is because it would be optimal to grant V more equity to provide sufficient incentives for V’s effort and investment than what makes V’s payoff equal the minimum value \( R \). Hence, it is better asking V to transfer some wealth \( T(\theta) \) to E in advance to ensure that V’s payoff is just \( R \) even though V gets such a higher share of equity. However, we do not consider the case where E transfers cash to V though it also would induce a Pareto improvement since E has no capital.
Maximizing their expected payoffs by FOC, we get the following Nash equilibrium of the game, called the second-best strategies:

**Proposition 3.2.** For a given contract defined by \( \{s(\theta), T(\theta)\}_{\theta \in \Theta} \) at the second stage, there is a Nash equilibrium of the game with the double-sided moral hazard problem between \( E \) and \( V \), given by strategy profile \( \{e^*(s(\theta), \theta); (a^*(s(\theta), \theta), k^*(s(\theta), \theta))\}_{\theta \in \Theta} \), where \( E \)'s strategy is

\[
e^*(s(\theta), \theta) = \left[ u(1 - \alpha_k - \alpha_a)(1 - s(\theta)) \right] \frac{2^{-\alpha_a - 2\alpha_k} (\alpha_a v)^{1 - \alpha_k} (\alpha_k v)^{1 - \alpha_a}}{2(1 - \alpha_k)(\alpha_k)^{\alpha_k} s(\theta)^{\frac{1 + \alpha_a + \alpha_k}{1 - \alpha_k}} (A\theta)^{\frac{1}{1 - \alpha_k}}},
\]

and \( V \)'s strategy is

\[
\begin{align*}
\{ & a^*(s(\theta), \theta) = \left[ u(1 - \alpha_k - \alpha_a)(1 - s(\theta)) \right] \frac{1 - \alpha_k - \alpha_a}{2(1 - \alpha_k)} (\alpha_a v)^{1 - \alpha_k} (\alpha_k v)^{1 - \alpha_a} \left( s(\theta) \right)^{\frac{1 + \alpha_a + \alpha_k}{1 - \alpha_k}} (A\theta)^{\frac{1}{1 - \alpha_k}}, \\
& k^*(s(\theta), \theta) = \left[ u(1 - \alpha_k - \alpha_a)(1 - s(\theta)) \right] \frac{1 - \alpha_k - \alpha_a}{2(1 - \alpha_k)} (\alpha_a v)^{1 - \alpha_k} (\alpha_k v)^{1 - \alpha_a} \left( s(\theta) \right)^{\frac{1 + \alpha_a + \alpha_k}{1 - \alpha_k}} (A\theta)^{\frac{1}{1 - \alpha_k}}. 
\end{align*}
\]

\( V \)'s equilibrium payoff is \( \Pi^SB_V(s(\theta), \theta) - T(\theta) \) and \( E \)'s equilibrium payoff is \( \Pi^SB_E(s(\theta), \theta) + T(\theta) \), where \( \Pi^SB_V(s(\theta), \theta) \) is given by \( (6) \) and \( \Pi^SB_E(s(\theta), \theta) \) is given by \( (7) \) with substitution of equilibrium strategy profile \( \{e^*(s(\theta), \theta); (a^*(s(\theta), \theta), k^*(s(\theta), \theta))\}_{\theta \in \Theta} \) for strategy profile \( \{e; a, k\} \).

We note that the equilibrium are independent of the transfer \( T(\theta) \). In fact, the transfer \( T(\theta) \) is designed only for making \( V \)'s payoff equal the minimum value \( R \).

**Optimal contract.** We aim to design an optimal contract \( \{s(\theta), T(\theta)\}_{\theta \in \Theta} \). To this end, we assume the contract is designed by \( E \) who manages to maximize her payoff subject to \( V \)'s participant constraint, \( E \)'s limited liability constraint and \( E \)'s liquidity constraint. Specifically, to design the optimal contract, \( E \) must fix the function pair \( \{s(\theta), T(\theta)\}_{\theta \in \Theta} \) that maximizes \( E \)'s equilibrium payoff, i.e., for a given profitability \( \theta \) revealed, \( E \) must solve the following optimization problem:

\[
\max_{s, T} \left\{ \Pi^SB_E(s, \theta) + T \right\} \\
\text{s.t.} \begin{cases} 
\Pi^SB_V(s, \theta) - T \geq R, \\
\Pi^SB_E(s, \theta) + T \geq 0, \\
T \geq 0,
\end{cases}
\]

\( \text{(12)} \)
where $\Pi_{SE}(s, \theta) + T$ is E’s equilibrium payoff and $\Pi_{SV}(s, \theta) - T$ is V’s equilibrium payoff given by Prop. 3.2 with substitution of $s$ and $T$ for $s(\theta)$ and $T(\theta)$ respectively.

To solve (12), the optimal share $s^*(\theta)$ of equity allocated to V is a solution of the following optimization problem:

$$\max_s \Pi_{SB}(s, \theta) - R \equiv Ak^*\alpha_k a^*\alpha_a e^*(1-\alpha_k-\alpha_a)\theta - \frac{e^2}{2u} - \frac{a^2}{2v} - k^* - R$$

s.t. $\Pi_{SV}(s, \theta) \geq R$

$$\Pi_{SB}(s, \theta) \geq R,$$  

(13)

where $e^*$ is given by (10) while $a^*$ and $k^*$ are given by (11) with suppression of the explicit dependence of $e^*$, $a^*$ and $k^*$ on $s$ and $\theta$. Then we get the optimal transfer $T^*(\theta) = \Pi_{SV}(s^*(\theta), \theta) - R$ since venture capital market is fully competitive as we assumed before. In equilibrium, we derive from Prop. 3.2 that at the beginning of the second stage, the second-best value of the project is

$$\Pi_{SB}(s, \theta)$$

$$= (\alpha_a v)^{\frac{\alpha_a}{1-\alpha_k}} [u(1 - \alpha_k - \alpha_a)(1 - s)]^{\frac{1-\alpha_k}{1-\alpha_k}} \alpha_k^{\frac{2a}{1-\alpha_k}} (s)^{\frac{\alpha_k + 2a}{1-\alpha_k}} (A\theta)^{\frac{2}{1-\alpha_k}}$$

$$\times \left( \frac{1}{2} + \frac{1}{2} \alpha_k + \frac{1}{2} \alpha_a + \frac{1}{2} s - \frac{3}{2} \alpha_k s - \alpha_a s \right),$$

(14)

and V’s value is

$$\Pi_{SV}(s, \theta)$$

$$= (\alpha_a v)^{\frac{\alpha_a}{1-\alpha_k}} [u(1 - \alpha_k - \alpha_a)(1 - s)]^{\frac{1-\alpha_k}{1-\alpha_k}} \alpha_k^{\frac{2a}{1-\alpha_k}} (s)^{\frac{1+\alpha_a + \alpha_k}{1-\alpha_k}} (A\theta)^{\frac{2}{1-\alpha_k}}$$

$$\times (1 - \frac{1}{2} \alpha_a - \alpha_k).$$

(15)

Accordingly, from $\Pi_{SE}(s, \theta) = \Pi_{SB}(s, \theta) - \Pi_{SV}(s, \theta)$ we conclude the following E’s value:

$$\Pi_{SE}(s, \theta)$$

$$= (\alpha_a v)^{\frac{\alpha_a}{1-\alpha_k}} [u(1 - \alpha_k - \alpha_a)(1 - s)]^{\frac{1-\alpha_k}{1-\alpha_k}} \alpha_k^{\frac{2 a}{1-\alpha_k}} s^{\frac{\alpha_a + 2 a}{1-\alpha_k}} (A\theta)^{\frac{2}{1-\alpha_k}}$$

$$\times \left( \frac{1}{2} + \frac{1}{2} \alpha_k + \frac{1}{2} \alpha_a - \frac{1}{2} s - \frac{1}{2} \alpha_k s - \frac{1}{2} \alpha_a s \right).$$

(16)

Thus we have $\Pi_{SE}(s, \theta) \geq 0$, implying that E’s participation constraint is satisfied.
Then we simplify the optimal contract problem as follows:

\[ s^*(\theta) = \arg\max_s \{ \Pi^{SB}(s, \theta) - R|\Pi^{SB}_V(s, \theta) \geq R \}, \quad (17) \]

while \( T^*(\theta) = \Pi_V(s^*(\theta), \theta) - R \). Noting that

\[
\frac{1}{2} + \frac{1}{2} \alpha_k + \frac{1}{2} \alpha_a + \frac{1}{2} s^*(\theta) - \frac{3}{2} \alpha_k s - \alpha_a s \\
= \frac{1}{2}(1 - s) + \frac{1}{2} \alpha_k(1 - s) + \frac{1}{2} \alpha_a + s(1 - \alpha_a - \alpha_k)
\]

> 0,

we have \( \Pi^{SB}(s, \theta) > 0 \), i.e. the project is profitable in equilibrium. We first consider the case where (17) has an interior solution. We conclude the following conclusion:

**Proposition 3.3.** If (17) has an interior solution, the optimal share \( s^{SB} \) of equity allocated to \( V \) does not depend on the profitability of the project and is given by

\[
s^{SB} = \begin{cases} 
\frac{1}{2}, & \text{if } 2\alpha_a + 3\alpha_k - 1 = 0, \\
\frac{(\alpha_a + 2\alpha_k)(1 + \alpha_a + \alpha_k) - \sqrt{\Delta}}{2(2\alpha_a + 3\alpha_k - 1)}, & \text{if } 2\alpha_a + 3\alpha_k - 1 \neq 0,
\end{cases}
\]

where

\[
\Delta = (\alpha_a + 2\alpha_k)(1 + \alpha_a + \alpha_k)[\alpha_a^2 + 3\alpha_a(\alpha_k - 1) + 2(\alpha_k - 1)^2] > 0.
\]

**Proof.** This is an optimization problem without constraints with the object function \( \Pi^{SB}(s, \theta) \), which is given by (14) with substitution of \( s \) for \( s^*(\theta) \). After tedious derivations, we find that there are two roots of the equation derived from FOC: One is uninteresting and the other is given by

\[
s = \begin{cases} 
\frac{1}{2}, & \text{if } 2\alpha_a + 3\alpha_k - 1 = 0, \\
\frac{(\alpha_a + 2\alpha_k)(1 + \alpha_a + \alpha_k) - \sqrt{\Delta}}{2(2\alpha_a + 3\alpha_k - 1)}, & \text{if } \Delta > 0 \\
1, & \text{if } \Delta \leq 0 \\
\frac{(\alpha_a + 2\alpha_k)(1 + \alpha_a + \alpha_k) - \sqrt{\Delta}}{2(2\alpha_a + 3\alpha_k - 1)}, & \text{if } \Delta > 0 \\
0, & \text{if } \Delta \leq 0
\end{cases}
\]

(19)
We have $\Delta > 0$ since $0 < \alpha_a < 1 - \alpha_k$. Hence, we immediately get (18) from (19).

If (17) has an interior solution, then $V$ must transfer wealth $T^*(\theta) > 0$ to $E$ in the contract according to our $E$’s strong bargaining position assumption. We can check that, even if players have hidden actions, allocating the share $s^{SB}$ to $V$ leads to a minimum social welfare loss. This good feature is due to the contractible transfer $T^*(\theta)$ from $V$ to $E$. Additionally, Equation (18) implies that the optimal share depends neither on profitability $\theta$ nor on effort cost factors $u$ or $v$ but only on the output elasticities of capital and labour. From the expected project income, we conclude that the total relative productivity (output elasticities) of $V$’s contributions (after taking both capital and effort input of $V$ into account) is $\alpha_a + 2\alpha_k$ and that of $E$’s is $1 - \alpha_a - \alpha_k$. Thus the first equality of (18) says that if their output elasticities are equal to each other, their equity shares are the same. This is consistent with economic intuition.

Clearly $s^{SB}$ is the best share delivered to $V$ if and only if $\Pi^{SB}_V(s^{SB}, \theta) \geq R$. Otherwise, if $\Pi^{SB}_V(s^{SB}, \theta) < R$, then $V$’s participation constraint is binding and the optimal contract $\{s^*(\theta), T^*(\theta)\}_{\theta \in \Theta}$ is given by $T^*(\theta) = 0$ and $s^*(\theta)$ satisfying $\Pi^{SB}_V(s^*(\theta), \theta) = R$. To derive a more explicit contract, we first derive the following conclusion:

**Lemma 3.4.** Letting $\hat{s}$ maximize $V$’s equilibrium payoff $\Pi^{SB}_V(s, \theta)$, then we have

$$\hat{s} \equiv \frac{(1 + \alpha_a + \alpha_k)}{2} > s^{SB},$$

where $s^{SB}$ is given by (18).

**Proof.** We conclude from (15) that

$$\frac{\partial \Pi^{SB}_V(s, \theta)}{\partial s} = (\alpha_a v)^{\frac{\alpha_a}{1 - \alpha_k}} [u(1 - \alpha_k - \alpha_a)]^{\frac{1 - \alpha_a - \alpha_k}{1 - \alpha_k}} \alpha_k^{\frac{2\alpha_k}{1 - \alpha_k}} s^{\frac{\alpha_a + 2\alpha_k}{1 - \alpha_k}} (1 - s)^{\frac{\alpha_k}{1 - \alpha_k}} (A^\theta)^{\frac{2}{1 - \alpha_k}} \left(1 - \frac{1}{2} \frac{\alpha_a - \alpha_k}{1 - \alpha_k}\right)[(1 + \alpha_a + \alpha_k) - 2s] = 0,$$

which yields the equality of (20). Next, we prove $\hat{s} > s^{SB}$. First, if $2\alpha_a + 3\alpha_k - 1 = 0$, we have $\frac{(1 + \alpha_a + \alpha_k)}{2} > \frac{1}{2}$. Second, since $0 \leq \alpha_a \leq 1$, $0 \leq \alpha_k \leq 1$ and $0 \leq \alpha_a + \alpha_k \leq 1$,
it is easy to prove \( \hat{s} > s^{SB} \) whether \( 2\alpha_a + 3\alpha_k - 1 > 0 \) or \( 2\alpha_a + 3\alpha_k - 1 < 0 \). So the lemma is proved.

Lemma 3.4 states that the share allocated to V that maximizes V’s equilibrium payoff depends on the sum of the output elasticity of V’s labour (effort) and that of V’s capital (investment cost \( k \)).

Figure 2. (a) E’s and V’s equilibrium payoff versus V’s share of equity when \( \alpha_a=0.4, \alpha_k=0.2, u = v = A = \theta=1 \), and (b) Optimal contract versus project profitability between E and V when \( \alpha_a=0.4, \alpha_k=0.2, u = v = A = R=1 \).

Fig. 2a shows the project value \( \Pi^{SB}(s, \theta) \) and V’s payoff \( \Pi^{SB}_V(s, \theta) \) in equilibrium under the second-best policies and the project value \( \Pi^{FB}(\theta) \) under the first-best policy. The share \( (s^{SB}) \) allocated to V that maximizes the project’s equilibrium payoff is less than the one \( (\hat{s}) \) that maximizes V’s value, as predicted by Lemma 3.4.

In the context of our study, \( \Pi^{SB}(s, \theta) \) is the function that needs to be maximized, subject to V’s participation constraint \( \Pi_V^{SB}(s, \theta) \geq R \). Therefore if \( \Pi_V(s^{SB}, \theta) \geq R \) holds, the optimal share given to V is \( s^{SB} \) and the transfer \( T(\theta) = \Pi^{SB}_V(s^{SB}, \theta) - R \). If \( \Pi_V^{SB}(s^{SB}, \theta) < R \leq \Pi_V^{SB}(\hat{s}, \theta) \) it follows that there exists an interval of shares of equity allocated to V included in the interval \([s^{SB}, \hat{s}]\) in which each share of equity allocated to V satisfies V’s participant constraint. This observation can be inferred from Fig. 2a, which shows that the optimal share \( s^*(\theta) \) of problem (17) is the smallest share of equity that makes V’s payoff greater than \( R \). In contrast, if \( \Pi_V^{SB}(\hat{s}, \theta) < R \), the project is not feasible since no share of equity allocated can
motivate V’s participating.\footnote{Please note that in contrast to V, E has no money transferred to V in advance as we assume before.}

To examine the influence of the profitability $\theta$ on the optimal contract, we know from (14) and (15) that $\Pi_{SB}(s, \theta)$ and $\Pi_{V}^{SB}(s, \theta)$ are increasing in $\theta$. Moreover, we have $\Pi_{V}^{SB}(s_{SB}, \theta) < \Pi_{V}(\hat{s}, \theta)$ and $\Pi_{SB}(s, \theta) > \Pi_{SB}(s_{SB}, \theta)$ for all $\theta > 0$. If $s \in (s_{SB}, \hat{s})$, $\Pi_{SB}(s, \theta)$ decreases and $\Pi_{V}^{SB}(s, \theta)$ increases with $s$. These conclusions are further explained by Fig. 2a. Formally, we get the following proposition.

**Proposition 3.5.** At the beginning of the second stage, the profitability $\theta$ is publicly known. Denote

$$
\theta_1 \equiv 1 - \frac{2R}{A} \left( \frac{2}{2 - \alpha_a - 2 \alpha_k} \right)^{1-\frac{\alpha_k}{2}} (\alpha_a v)^{-\frac{\alpha_k}{2}} \left[ u(1 - \alpha_k - \alpha_a)^2 / 2 \right]^{\frac{1-\alpha_k-\alpha_a}{2}} \\
\times \alpha_k \left( 1 + \alpha_k + \alpha_a \right)^{-\frac{1+\alpha_k+\alpha_a}{2}}
$$

$$
\theta_2 \equiv 1 - \frac{2R}{A} \left( \frac{2}{2 - \alpha_a - 2 \alpha_k} \right)^{1-\frac{\alpha_k}{2}} (\alpha_a v)^{-\frac{\alpha_k}{2}} \left[ u(1 - \alpha_k - \alpha_a)(1 - s_{SB}) \right]^{\frac{1-\alpha_k-\alpha_a}{2}} \\
\times \alpha_k s_{SB}^{-\frac{1+\alpha_k+\alpha_a}{2}},
$$

where $s_{SB}$ is defined in (18).\footnote{To focus on our problem we assume $\theta_1, \theta_2 \in [\theta_L, \theta_H]$.}

Then the optimal contract $\{s^*(\theta), T^*(\theta)\}_{\theta \in \Theta}$ has the following form: If $\theta < \theta_1$, the second-stage investment is worthless; If $\theta_1 \leq \theta < \theta_2$, $s^*(\theta) = \min \{ s | \Pi_{V}^{SB}(s, \theta) = R \}$ and $T^*(\theta) = 0$, $s^*(\theta) \in (s_{SB}, \hat{s}]$ and $(s^*(\theta))' < 0$; If $\theta \geq \theta_2$, $s^*(\theta) = s_{SB}$ and $T^*(\theta) = \Pi_{V}^{SB}(s_{SB}, \theta) - R$.

**Proof.** Noting that $\theta_1$ is the minimum profitability ensuring V’s participation we have $\Pi_{V}^{SB}(\hat{s}, \theta_1) = R$, and thus $\theta_1$ is derived. Naturally, if $\theta < \theta_1$, V will not participate in the contract.

The threshold $\theta_2$ is derived from the equality

$$
\Pi_{V}^{SB}(s_{SB}, \theta_2) = R.
$$

As a consequence, we have $\theta_1 < \theta_2$, since $\Pi_{V}^{SB}(s_{SB}, \theta_2) = \Pi_{V}^{SB}(\hat{s}, \theta_1) > \Pi_{V}^{SB}(s_{SB}, \theta_1)$.

If $\theta_1 \leq \theta < \theta_2$, noting that $\frac{\partial \Pi_{V}^{SB}(s, \theta)}{\partial \theta} > 0$ and $s = s(\theta) \leq \hat{s}$, we have $\frac{\partial \Pi_{V}^{SB}(s, \theta)}{\partial s} > 0$.
0. To keep \( \Pi_{SB}^{V}(s, \theta) = R \), it can be inferred that \( s'(\theta) < 0 \), indicating that the required share for \( V \) decrease as profitability \( \theta \) increases. It leads to \( s^*(\theta) \in (s^{SB}, \hat{s}] \). Specifically, we have \( s^*(\theta) = \min \{ s | \Pi_{SB}^{V}(s, \theta) = R \} \) thanks to the monotonicity of \( \Pi_{SB}(s, \theta) \) and \( \Pi_{SB}^{V}(s, \theta) \) within this interval.

Fig. 2b displays how the optimal share \( s^*(\theta) \) and optimal transfer payment \( T^*(\theta) \) change with the revealed profitability \( \theta \) in the optimal contract. As predicted by Prop. 3.5, the figure says that the optimal shares \( (s^*(\theta)) \) are decreasing in \( \theta \) and the transfer \( (T^*(\theta)) \) is zero when the revealed profitability is less favourable \( (\theta_1 < \theta < \theta_2) \). If the revealed profitability exceeds a threshold \( (\theta_2) \), the optimal share given to \( V \) is a fixed constant \( s^{SB} \), and the transfer payment \( T^*(\theta) \) increases quickly with \( \theta \).

Social welfare loss due to moral hazard. We compare the social surplus under full information (first-best case) with that under moral hazard (second-bast case) to obtain the welfare loss rate caused by moral hazard. Naturally, the loss rate \( l(\theta) \) is defined by\(^6\)

\[
l(\theta) = 1 - \frac{\Pi_{SB}^{V}(s^*(\theta), \theta)}{\Pi_{FB}^{V}(\theta)}.\]

Substituting (5) and (14) into it yields

\[
l(\theta) = 1 - \left(1 - s^*(\theta)\right)^{\frac{1-\alpha_k-\alpha_a}{(1-\alpha_k)}} \left(s^*(\theta)\right)^{\frac{\alpha_k+2\alpha_a}{(1-\alpha_k)}} \frac{1 + \alpha_k + \alpha_a + (1 - 3\alpha_k - 2\alpha_a)s^*(\theta)}{1 - \alpha_k} \cdot f(s^*(\theta)) \cdot \frac{1}{(1-3\alpha_k-2\alpha_a)s^{SB}}. \tag{23}\]

We find that as we expected by intuition, if \( V \)'s participation constraint is not binding, i.e. the optimal share allocated to \( V \) is \( s^{SB} \), the welfare loss rate arrives at its minimum value since we can prove \( \frac{df(s^{SB})}{ds} = 0 \). To be specific, we assume the project is sufficient profitable and participation constraints are satisfied in discussing the welfare loss rate below.

Plugging \( s^*(\theta) = s^{SB} \) into (23) gives

\[
l = 1 - \left(1 - s^{SB}\right)^{\frac{1-\alpha_k-\alpha_a}{1-\alpha_k}} \left(s^{SB}\right)^{\frac{\alpha_k+2\alpha_a}{1-\alpha_k}} \frac{1 + \alpha_k + \alpha_a + (1 - 3\alpha_k - 2\alpha_a)s^{SB}}{1 - \alpha_k},\]

\(^6\)For simplicity, we assume here that \( V \)'s participant constraint is not binding, i.e. the revealed profitability \( \theta \geq \theta_2 \).
where $s^{SB}$ is given by (18), indicating that the welfare loss rate $l$ arising from moral hazard depends only on inputs’ output elasticities $(\alpha_a, \alpha_k)$. We emphasize that this result holds even the optimal contract is signed at the very beginning as discussed in Section 5, where the initial investment has not been exercised.

Fig. 3 shows the loss rate caused by moral hazard under different inputs’ output elasticities. There is an inverse U-shaped relationship between the welfare loss rate and output elasticities. Social welfare loss increases first and then decreases with the elasticity of effort (labour) output, as well as that of capital output. When $E$ or $V$ completely control the output ($\alpha_a + \alpha_k = 1$ or $\alpha_a = \alpha_k = 0$), indicating that a two-player game degenerates into a single one, the welfare loss arrives at the minimum value zero. This is quite in agreement with economic intuition. Roughly speaking, the more evenly matched the participants’ role, the higher the welfare loss rate. Specifically, when capital’s output elasticity $\alpha_k = 0.2$, the largest loss rate is realized if the output elasticity of $V$’s effort $\alpha_a = 0.2$. And when $\alpha_a = 0.4$, the largest loss rate is realized for $\alpha_k = 0.34$.

In summary, the optimal contract analysed in this section provides a basis for later analysing the two-stage investment. Our analysis begins by illustrating the first-best strategies with full information. Subsequently, we delve into the impact of the revealed profitability on incentives in a principal-agent framework, where $V$ contributes both his advice and capital and $E$ provides her effort. When the revealed profitability is favourable enough ($\theta \geq \theta_2$), the equity split is solely dependent on
their output elasticities ($\alpha_a$, $\alpha_k$), leading to the lowest loss rate of social welfare. Conversely, in the case of somewhat unfavourable prospects ($\theta_1 \leq \theta < \theta_2$), V’s participation constraint is binding, resulting in an upward distortion of the optimal share. This distorted share is highly sensitive to the V’s opportunity cost $R$ and decreases in the revealed profitability $\theta$. Furthermore, the reduction in social welfare in this scenario becomes increasingly severe as the share $s$ increases.

Next, we consider the optimal contract prior to investment where the profitability is not revealed. We consider two cases: one is that the potential profitability is verifiable with non-negotiable contract, and the other is that the profitability is observable but not verifiable with renegotiation.

4. Optimal contract with two-stage investment if project profitability is verifiable

Armed with second-stage investment conclusions derived in the last section, we turn to the first-stage investment. In this section, we consider a simple situation where profitability $\theta$ is verified or equivalently contractible. Prior to investment, E and V sign a long-term contract, in which V pays investment cost $k_0$ in the first stage, aiming to explore project profitability. After the profitability is revealed, in the second stage, E and V freely exert their efforts, and particularly, V can flexibly determine whether and how much to continue the subsequent investment according to the revealed profitability $\theta$. However, the share of equity allocated to V, which is a function of $\theta$, is specified at the starting time.

Specifically, the contract is defined by a triple $\{T, \underline{\theta}, \{z(\theta)\}_{\theta \in \Theta}\}$ where $\underline{\theta} \geq \theta_L$ is the cut-off point of the project profitability below which the project must be abandoned since it is worthless although V has invested in $k_0$, $z(\theta) \in (0, 1)$ is the share allocated to V if profitability $\theta \geq \underline{\theta}$, and an initial deterministic transfer $T \geq 0$ that E receives from V. As discussed in the last section, the transfer would be necessary to maximize E’s value while V gets $R + k_0$, the minimum value that makes V willing to enter into the contract.

E’s problem is to design an optimal contract $\{T^*, \underline{\theta}^*, \{z^*(\theta)\}_{\theta \in \Theta}\}$, such that E’s expected payoff is maximized subject to E’s and V’s participant constraint. Thanks
to previous equilibrium payoffs, E needs to solve the following optimization problem:\(^7\)

\[
\max_{T, \theta, \{z(\theta)\}_{\theta \in \Theta}} \left\{ \mathbb{E}[\Pi^E_S(z(\theta), \theta)] + T = \int_{\theta_H}^{\theta_H} \Pi^E_S(z(\theta), \theta)dF(\theta) + T \right\},
\]

subject to

\[
\begin{align*}
\mathbb{E}[\Pi^V_S(z(\theta), \theta)] - T &= \int_{\theta_H}^{\theta_H} \Pi^V_S(z(\theta), \theta)dF(\theta) - T \\
\mathbb{E}[\Pi^E_S(z(\theta), \theta)] + T &= \int_{\theta_H}^{\theta_H} \Pi^E_S(z(\theta), \theta)dF(\theta) + T \geq 0, \quad (E's \ limited \ liability) \\
T &\geq 0, \quad (E's \ wealth \ constraint: \ E \ has \ no \ capital \ at \ first.)
\end{align*}
\]

where \( F(\theta) \) represents the cumulative distribution function of the random profitability \( \theta \). As before, V's participation constraint will be satisfied with equality and the participation constraint of E is not binding, and thus we rewrite the problem in the following equivalent form,

\[
\max_{z(\theta), \theta} \int_{\theta_H}^{\theta_H} \Pi^S_B(z(\theta), \theta)dF(\theta) - R - k_0,
\]

subject to

\[
\int_{\theta_H}^{\theta_H} \Pi^S_B(z(\theta), \theta)dF(\theta) - R - k_0 \geq 0.
\]

First, we characterize two thresholds: One is the maximum initial investment cost \( H_0 \) that makes the project feasible and the other is the minimum initial investment cost \( L_0 \) that makes the transfer from V to E unnecessary. We have

**Lemma 4.1.** Let

\[
\begin{align*}
H_0 &\equiv \max_{z(\theta), \theta} \int_{\theta_H}^{\theta_H} \Pi^S_B(z(\theta), \theta)dF(\theta) - R = \int_{\theta_L}^{\theta_H} \Pi^S_B(\hat{s}, \theta)dF(\theta) - R, \\
L_0 &\equiv \int_{\theta_H}^{\theta_H} \Pi^S_B(s^{SB}, \theta)dF(\theta) - R = \int_{\theta_L}^{\theta_H} \Pi^S_B(s^{SB}, \theta)dF(\theta) - R,
\end{align*}
\]

and then we have: (i) the project is infeasible if \( k_0 > H_0 \); (ii) V invests \( k_0 \) without paying a transfer if \( L_0 < k_0 \leq H_0 \); (iii) V invests \( k_0 \) and gives an initial transfer of

---

\(^7\)As usual in mathematical finance, we assume the time discount rate is zero throughout the text since the interest rate is not important. Moreover, for ease to implement the contract, we do not consider the possibility of transferring a wealth from V to E in the second stage to maximize E’s value as discussed in the last section.
\( T = L_0 - k_0 \) to \( E \) at the same time if \( k_0 \leq L_0 \); (iv) \( V \) will never abandon the project \((\theta = \theta_L)^8\)

**Proof.** According to Lemma 3.4, we conclude that \( \max_s \Pi^{SB}_V(s, \theta) = \Pi^{SB}_V(\hat{s}, \theta) \). From (15), we get \( \Pi^{SB}_V(s, \theta) > 0 \) for all \( \theta \in \Theta \), then \( \theta = \theta_L \) is proved.\(^9\) It is direct that when the initial investment exceeds the difference between \( V \)’s maximum expected return and opportunity cost, \( V \) will not invest in the first place. When the initial investment is lower than the difference between \( V \)’s return under globally optimal share \((s^{SB})\) and opportunity cost, \( V \)’s participation constraint will not be binding. \( \square \)

Lemma 4.1 gives the condition of whether to invest in \( k_0 \) at startup stage and demonstrates \( V \) will always invest in the expansion stage. Specifically, only if the maximum expected return is sufficient to cover \( k_0 \) and the opportunity cost \( R \), will \( V \) invest in the startup stage. And once \( V \) invests in the startup stage, he will certainly continue to invest in the expansion stage. So the contract \( \{T, \theta, \{z(\theta)\}_{\theta \in \Theta}\} \) comes down to \( \{T, \{z(\theta)\}_{\theta \in \Theta}\} \). Next, we analyse the optimal contract with verifiable profitability information.

**Proposition 4.2.** When the information on the project profitability \( \theta \) is verifiable, the optimal contract is characterized as follows:

(i) If \( k_0 > H_0 \), the project is not feasible;

(ii) If \( L_0 < k_0 \leq H_0 \) the share allocated to \( V \) is a constant share \( z(\theta) = \bar{s} \in (s^{SB}, \hat{s}] \) satisfying \( \bar{s} = \min \{s, \int_{\theta_L}^{\theta_H} \Pi^{SB}_V(s, \theta)d\Phi(\theta) - R - k_0 = 0\} \), for all \( \theta \in \Theta \), and the transfer \( T = 0 \) from \( V \) to \( E \);

(iii) If \( k_0 \leq L_0 \) the share of equity allocated to \( V \) is a constant share \( z(\theta) = s^{SB} \), for all \( \theta \in \Theta \), and the transfer \( T = L_0 - k_0 \) from \( V \) to \( E \).

**Proof.** According to the definition of \( H_0 \) in Lemma 4.1, \( V \)’s participation constraint is not satisfied if \( k_0 > H_0 \), and so part (i) is proved. If \( k_0 \leq H_0 \), by the saddle point theorem, problem (25) reduces to the following form:

\[
\max_{z(\theta)} \int_{\theta_L}^{\theta_H} \Pi^{SB}_V(z(\theta), \theta)d\Phi(\theta) - R - k_0 + \gamma_0 \left( \int_{\theta_L}^{\theta_H} \Pi^{SB}_V(z(\theta), \theta)d\Phi(\theta) - R - k_0 \right),
\]

\( (27) \)

---

\(^8\)Combine the previous assumptions and conclusions, we have \( 0 \leq \theta_L = \theta \leq \theta_1 < \theta_2 \leq \theta_H \).

\(^9\)It can also be proved in Prop. 4.2 with non-linear programming methods.
where \( \gamma_0 \geq 0 \) is the Lagrange multiplier. Applying Kuhn-Tucker theorem gives

\[
\begin{align*}
\frac{\partial \Pi^{SB}_V(z(\theta),\theta)}{\partial s} + \gamma_0 \frac{\partial \Pi^{SB}_V(z(\theta),\theta)}{\partial s} &= 0, \\
\gamma_0 (\int_{\theta_L}^{\theta_H} \Pi^{SB}_V(z(\theta),\theta)dF(\theta) - R - k_0) &= 0, \\
\gamma_0 &\geq 0, \\
-\Pi^{SB}_V(z(\theta),\theta) - \gamma_0 \Pi^{SB}_V(z(\theta),\theta) &< 0.
\end{align*}
\]

(28)

Since the Lagrangian decreases with \( \theta \), no matter whether \( \gamma_0 = 0 \) or \( \gamma_0 > 0 \), we have \( \theta = \theta_L \) as proved by Lemma 4.1.

If the constraint is not binding \( (k_0 \leq L_0) \), we get \( \gamma_0 = 0 \), and

\[
\frac{\partial \Pi^{SB}_V(z(\theta),\theta)}{\partial s} = 0.
\]

Thus, we have \( z(\theta) = s^{SB} \) by Prop. 3.3. V should give a transfer \( T = L_0 - k_0 \) to E at the beginning of the startup stage.

If the constraint is binding, where \( H_0 \geq k_0 > L_0 \), we get \( \gamma_0 > 0 \) and

\[
\frac{\partial \Pi^{SB}_V(z(\theta),\theta)}{\partial s} \left( \frac{\partial \Pi^{SB}_V(z(\theta),\theta)}{\partial s} \right)^{-1} = -\gamma_0.
\]

(29)

Substituting (14) and (15) into this equation, we know that the left-hand side (LHS hereafter) of (29) does not depend on \( \theta \), so there is a constant share \( z(\theta) = \bar{s} \) solving (29). Since \( \frac{\partial \Pi^{SB}_V(z(\theta),\theta)}{\partial s} < 0 \) if and only if \( s > s^{SB} \), and \( \frac{\partial \Pi^{SB}_V(z(\theta),\theta)}{\partial s} > 0 \) if and only if \( s < \hat{s} \), we conclude \( s^{SB} < \bar{s} < \hat{s} \). We also derive that \( \bar{s} \) is a solution of the following equation:

\[
\int_{\theta_L}^{\theta_H} \Pi^{SB}_V(\bar{s},\theta)dF(\theta) - R - k_0 = 0.
\]

As in Prop. 3.5, we take the smallest \( \bar{s} \) that satisfies this condition. \( \square \)

Prop. 4.2 provides a comprehensive analysis of the optimal contract in scenarios where project profitability is contractible. The results indicate that if the initial investment cost \( k_0 \) exceeds a certain threshold \( (H_0) \), i.e. the difference between the V’s maximum expected payoff and his opportunity cost, investing in the project is infeasible. By contrast, if the investment cost is sufficiently low \( (k_0 \leq L_0) \), the
optimal share $s^{SB}$ can be taken to maximize E’s expected payoff, with an initial transfer $T = L_0 - k_0$ from V. For intermediate values of $k_0$ ($L_0 < k_0 \leq H_0$), investing in the project is feasible, and V will require a constant share $\bar{s}$ without a transfer, where $\bar{s}$ is distorted from $s^{SB}$ and close to $\hat{s}$.

Fig. 4a displays how the optimal V’s share and the transfer change with the initial investment cost $k_0$ in the optimal contract. It says that if $L_0 < k_0 \leq H_0$, the optimal V’s share $z$ is increasing with $k_0$, even though it is independent of $\theta$.

![Graph showing the contract vs. initial cost and V’s share vs. profitability](image)

(a) The contract vs. initial cost   (b) V’s share vs. profitability

**Figure 4.** (a) The optimal contract changes with the initial cost, and (b) V’s optimal share varies with the verifiable profitability when $k_0=0.02$, where $\alpha_a=0.4$, $\alpha_k=0.2$, $\theta_L=3.2$, $\theta_H=5.2$, $u = v = A = R = 1$.

Note that under the contingent contract, once the type of contract in Prop. 4.2 is initially determined according to the relative value of $k_0$, the contract will be strictly executed, even if it does not conform to expectations when the intermediate information on $\theta$ is revealed later. For example, when $\theta > \theta_2$, V will still obtain a larger share $\bar{s}$ ($\bar{s} > s^{SB}$).

Fig. 4b shows how the optimal share $z(\theta)$ here and $s^*(\theta)$ derived in the last section variate with project profitability ($\theta$) if initial investment cost satisfies $L_0 < k_0 \leq H_0$. It states that when V’s participation constraint is binding, the optimal share specified in Prop. 3.5 decreases with $\theta$ while that in Prop. 4.2 is constant. The optimal contract in Prop. 4.2 enables the project to continue even if the net payoff for the V is negative, provided that $L_0 < k_0 + R < H_0$ and $\theta < \theta < \theta_1$, indicating that venture capital investment resembles a gamble when the information is verifiable. V decides whether to participate in the investment and accept the corresponding
contract based on the expected payoff. V will continue to invest even though his net
present value is negative since he has promised to keep investing for the expansion
period as long as $\theta$ is greater than $\theta$. On the contrary, once the profitability revealed
exceeds the expectation, E must also keep her promise to give V a higher share and V
will get excess remuneration. In this case, it increases the possibility for cooperation
in low-profitability scenarios by allowing high-profitability states to subsidize some
of the low-profitability states.

5. Optimal contract in two-stage investment with renegotiation and ob-
    servable but unverifiable profitability

It is possible that project profitability is observable but unverifiable or not con-
tractible, where we cannot specify V’s share depending on the profitability before
it is revealed. Thus, V’s share, denoted by $s$ instead of $z(\theta)$ in the last section, as
well as a transfer $T$ from V to E must be constant in a contract. Naturally, the
previous specified V’s share $s$ would be not a best allocation after project profita-
blility is revealed, and hence there is a Pareto improvement via renegotiation of equity
split. To be specific, we further assume that E has all the bargaining power, and the
renegotiation surplus is totally harvested by E in the following text.

The contract is defined by a pair $\{s, T\}$ specifying (a) V’s share $s \in (0, 1)$ of
the project’s success income $\theta$ ($\theta \in \Theta$), and (b) an initial transfer $T \geq 0$ that E
receives from V. At first sight, it seems that in renegotiation, we could increase V’s
share with a transfer from V to E meanwhile to increase E’s value while keeping
V’s value unchanged. Actually, given a contract defined by $\{s, T\}$, if $s = s^{SB}$, then
the equity allocation is already Pareto-optimal, and thus any renegotiation will not
make a Pareto improvement. We will see later that increasing V’s share is a possible
Pareto improvement in renegotiation but it does not induce wealth transfer since the
aim of increasing V’s share is just to satisfy V’s participant constraint in the second
stage.

We consider two cases in the following: One assumes the transfer $T$ received by E
from V after entering into the contract cannot be rolled back to V in renegotiation;
The other is that the rollback is permissible. We use a backward induction method.
We begin by analysing conditions under which would the two sides renegotiate and derive the renegotiation results defined by a pair \(\{s_{n2}(\theta), \tilde{T}(\theta)\}_{\theta \in \Theta}\) where \(s_{n2}(\theta)\) indicates the new share allocated to V and \(\tilde{T}(\theta)\) represents the possible transfer from E to V. After that, we derive the optimal contract signed in the initial time.

5.1. Optimal contract with renegotiation if initial transfer cannot be rolled back

We assume both E and V know that the contract would be renegotiated. First, for a given contract defined by \(\{s, T\}\), as explained by Fig. 2a, we can prove that it is not optimal if \(s < s^{SB}\) or \(s > \hat{s}\) due to the fact that E has no money to pay V in the first-stage investment.

Second, since we assume here that the transfer \(T\) cannot be rolled back, we cannot decrease V’s share in renegotiation. Therefore we focus on the case of \(s^{SB} \leq s \leq \hat{s}\) and the possible Pareto improvement by increasing V’s share in the following renegotiation. We define the threshold \(\theta_1\) that satisfies \(\Pi_{SB}^{V}(\hat{s}, \theta_1) = R\).

Then we have

**Lemma 5.1.** For a given contract defined by \(\{s, T\}\) with \(s^{SB} \leq s \leq \hat{s}\), V abandons the project if the revealed profitability \(\theta \leq \theta_1\). Otherwise, the renegotiation is necessary only when V’s participant constraint is not satisfied, and the new share allocated to V after renegotiation is the minimum share that satisfies V’s participation constraint in the second stage.

**Proof.** Letting \(s \in [s^{SB}, \hat{s}]\), we have

\[
\frac{\partial \Pi_{SB}^{V}(s, \theta)}{\partial s} < 0 \quad \text{and} \quad \frac{\partial \Pi_{SB}^{V}(s, \theta)}{\partial s} > 0.
\]

Therefore, if the revealed profitability \(\theta \geq \theta_1\) and \(\Pi_{SB}^{V}(s, \theta) < R\), there is a minimum share \(s < s^*(\theta) \leq \hat{s}\) that satisfies V’s participation constraint in the second stage while maximizing E’s value. Here \(s^*(\theta)\), given by Prop.3.5, is the new share after renegotiation. In addition, if \(\Pi_{SB}^{V}(s, \theta) \geq R\), then no Pareto improvement will be realized since we assume rollback is prohibited. Finally, if \(\theta < \theta_1\), no matter how much the new share is, V’s participant constraint is not satisfied, indicating that the second-stage investment will not be exercised. \(\Box\)
The lemma asserts that the renegotiation will only occur if V is unwilling to continue the investment under the original contract \( \{s, T\} \). Referring to (16), E’s return is always positive, E is willing to increase V’s share if and only if V is unwilling to continue investing. Since \( k_0 \) is the sunk cost having been paid at the time, E is only willing to raise V’s share \( s \) to the new share \( s^*(\theta) \) (at most \( \hat{s} \)) only for meeting V’s participation constraint, and more share will decrease E’s welfare.

Based on Lemma 5.1, we conclude that after renegotiation, V’s new share \( s_n(\theta) \) is given by

\[
s_n(\theta) = \begin{cases} 
  s, & \text{if } \theta_H \geq \theta \geq \theta^{**}, \\
  s^*(\theta), & \text{if } \theta^{**} \geq \theta \geq \theta_1, \\
  \emptyset, & \text{otherwise},
\end{cases}
\]

where \( \emptyset \) indicates that V will abandon the project when \( \theta < \theta_1 \), \( s^*(\theta) \) is given by Prop. 3.5 and \( \theta^{**} \) is defined by \( \Pi_{SV}^E(\theta^{**}) = R \), that is

\[
\theta^{**} = \frac{1}{A} \left( \frac{2R}{2 - \alpha_a - 2\alpha_k} \right)^{\frac{1-\alpha_k}{2}} (\alpha_a v)^{-\frac{\alpha_a}{2}} \left[ u(1 - \alpha_k - \alpha_a)(1 - s) \right]^{-\frac{1-\alpha_k}{2}} \alpha_k^{-\frac{1+\alpha_k}{2}} s^{-\frac{1+\alpha_k}{2}}.
\]

Now we turn to the optimal contract defined by \( (s^{**}, T^*) \). Noting that project profitability \( \theta \) is not observed when the contract is signed, we conclude that the optimal contract \( (s^{**}, T^*) \) solves the following optimization problem:

\[
\begin{aligned}
\max_{s,T} \left\{ \int_{\theta^*}^{\theta_H} 0dF(\theta) + \int_{\theta^*}^{\theta^{**}} \Pi_{SV}^E(s^*(\theta), \theta)dF(\theta) + \int_{\theta^{**}}^{\theta_H} \Pi_{SV}^E(s, \theta)dF(\theta) + T \right\} \\
\text{s.t.} \begin{cases} 
  \int_{\theta^*}^{\theta_H} 0dF(\theta) + \int_{\theta^*}^{\theta^{**}} \Pi_{SV}^E(s^*(\theta), \theta)dF(\theta) + \int_{\theta^{**}}^{\theta_H} \Pi_{SV}^E(s, \theta)dF(\theta) = T \geq R + k_0, \\
  T \geq 0.
\end{cases}
\end{aligned}
\]

As did in the last section, the optimal equity split problem can be restated as that \( s^{**} \) maximizes the following expected net present value of the project over \( s \):

\[
\begin{aligned}
\max_s \left\{ \int_{\theta^*}^{\theta_H} 0dF(\theta) + \int_{\theta^*}^{\theta^{**}} \Pi_{SV}^E(s^*(\theta), \theta)dF(\theta) + \int_{\theta^{**}}^{\theta_H} \Pi_{SV}^E(s, \theta)dF(\theta) - R - k_0 \right\} \\
\text{s.t.} \begin{cases} 
  \int_{\theta^*}^{\theta_H} 0dF(\theta) + \int_{\theta^*}^{\theta^{**}} \Pi_{SV}^E(s^*(\theta), \theta)dF(\theta) + \int_{\theta^{**}}^{\theta_H} \Pi_{SV}^E(s, \theta)dF(\theta) - R \geq k_0.
\end{cases}
\end{aligned}
\]

The following result characterizes the solution to this problem.
Proposition 5.2. Suppose that E’s transfer received from V cannot be rolled back to V in renegotiation. Denoting

\[ H_1 \equiv \int_{\tilde{\theta}_1}^{\tilde{\theta}_2} \Pi^{\text{SB}}(\tilde{s}, \theta)dF(\theta) - R \leq H_0, \quad L_1 \equiv \int_{\tilde{\theta}_1}^{\tilde{\theta}_2} R\theta dF(\theta) + \int_{\tilde{\theta}_2}^{\theta_H} \Pi^{\text{SB}}(s^{\text{SB}}, \theta)dF(\theta) - R, \]

we have the following optimal contract \( \{ s^{**}, T^* \} \): (i) if \( k_0 > H_1 \), the project is not feasible; (ii) if \( L_1 < k_0 \leq H_1 \), \( T^* = 0 \) and \( s^{**} = \tilde{s} \) where constant \( \tilde{s} \in (s^{\text{SB}}, \hat{s}) \) satisfies \( \tilde{s} = \min \{ \tilde{s} \mid \int_{\theta_1}^{\tilde{\theta}} R\theta dF(\theta) + \int_{\tilde{\theta}}^{\theta_H} \Pi^{\text{SB}}(\tilde{s}, \theta)dF(\theta) - R - k_0 = 0 \} \), and \( \tilde{\theta} \) satisfies \( \Pi^{\text{SB}}(\tilde{s}, \tilde{\theta}) = R \); (iii) if \( k_0 \leq L_1 \), \( s^{**} = s^{\text{SB}} \) and \( T^* = L_1 - k_0 \).

Proof. According to Lemma 3.4, we have max \( \Pi^{\text{SB}}_V(s, \theta) = \Pi^{\text{SB}}_V(\hat{s}, \theta) \) and \( \Pi^{\text{SB}}_V(s, \theta) \geq R \) if and only if \( \theta \geq \theta_1 \). Therefore, V’s payoff is maximized by setting \( s^{**} = \hat{s} \) and thus \( \theta^{**} = \theta_1 \). By the definition of \( H_1 \), it is direct that V’s participation constraint is never satisfied if \( k_0 > H_1 \), i.e. the project is not feasible.

If \( k_0 \leq H_1 \), the Lagrangian \( \tilde{L} \) of the problem is

\[
\tilde{L} = \int_{\theta_1}^{\theta^{**}} \Pi^{\text{SB}}(s^*(\theta), \theta)dF(\theta) + \int_{\theta^{**}}^{\theta_H} \Pi^{\text{SB}}(s, \theta)dF(\theta) - R - k_0
\]

\[
+ \gamma_1 \left( \int_{\theta_1}^{\theta^{**}} \Pi^{\text{SB}}_V(s^*(\theta), \theta)dF(\theta) + \int_{\theta^{**}}^{\theta_H} \Pi^{\text{SB}}_V(s, \theta)dF(\theta) - R - k_0 \right), \quad (32)
\]

where \( \gamma_1 \) is a non-negative Lagrange multiplier. Thanks to Kuhn-Tucher Theorem, we get

\[
\begin{aligned}
\Pi^{\text{SB}}(s^*(\theta), \theta^{**}) \frac{d\theta^{**}}{ds} - \Pi^{\text{SB}}(s^{**}, \theta^{**}) \frac{d\theta^{**}}{ds} + \int_{\theta^{**}}^{\theta_H} \frac{d\Pi^{\text{SB}}_V(\theta^*, \theta)}{d\theta} dF(\theta) \\
+ \gamma_1 \left( \Pi^{\text{SB}}_V(s^*(\theta), \theta^{**}) \frac{d\theta^{**}}{ds} - \Pi^{\text{SB}}_V(s^{**}, \theta^{**}) \frac{d\theta^{**}}{ds} + \int_{\theta^{**}}^{\theta_H} \frac{d\Pi^{\text{SB}}_V(\theta^*, \theta)}{d\theta} dF(\theta) \right) = 0,
\end{aligned}
\]

\[
\gamma_1 \left( \int_{\theta_1}^{\theta^{**}} \Pi^{\text{SB}}_V(s^*(\theta), \theta)dF(\theta) + \int_{\theta^{**}}^{\theta_H} \Pi^{\text{SB}}_V(s^{**}, \theta)dF(\theta) - R - k_0 \right) = 0,
\]

\( \gamma_1 \geq 0. \)

If the constraint is not binding (\( k_0 \leq L_1 \)), we get \( \gamma_1 = 0 \), and \( \Pi^{\text{SB}}(s^*(\theta), \theta^{**}) = \Pi^{\text{SB}}(s^{**}, \theta^{**}) \) by the definition of \( s^*(\theta) \). Thus \( \frac{d\Pi^{\text{SB}}_V(\theta^*, \theta)}{d\theta} = 0 \) leads to \( s^{**} = s^{\text{SB}} \) by Prop. (3.3). Since the participation constraint is not binding, V gives a transfer \( T^* = \int_{\theta_1}^{\theta_2} \Pi^{\text{SB}}_V(s^*(\theta), \theta)dF(\theta) + \int_{\theta_2}^{\theta_H} \Pi^{\text{SB}}_V(s^{SB}, \theta)dF(\theta) - R - k_0 = L_1 - k_0 \) to E at the beginning of the startup stage.
If the participation constraint is binding, where \( H_1 \geq k_0 > L_1 \), we get \( \gamma_1 > 0 \).
This implies \( \int_{\theta^*}^{\theta^*_{**}} \frac{d\Pi^B(s, \theta)}{ds} dF(\theta) + \gamma_1 (\int_{\theta^*}^{\theta^*_{**}} \frac{d\Pi^B(s, \theta)}{ds} dF(\theta)) = 0 \). After a careful derivation, we get
\[
\frac{d\Pi^B(s, \theta)}{ds} \left( \frac{d\Pi^B(s, \theta)}{ds} \right)^{-1} = -\gamma_1. (34)
\]
Substituting (14) and (21) into it, we get that the LHS actually does not depend on \( \theta \), and there is a constant share \( \tilde{s} \) satisfying
\[
\int_{\tilde{\theta}}^\theta RdF(\theta) + \int_{\tilde{\theta}}^{\theta} \Pi_S^B(s, \theta)dF(\theta) - R - k_0 = 0,
\]
where \( \tilde{\theta} \) satisfies \( \Pi_S^B(s, \tilde{\theta}) = R \), leading to
\[
\tilde{\theta} \equiv \frac{1}{A \left( 2 - \alpha_a - 2\alpha_k \right)} \left( \alpha_a v \right)^{-\alpha_k^2 / 2} \left[ u(1 - \alpha_k - \alpha_a)(1 - \tilde{s}) \right]^{-1 / 2}
\]
\[
\alpha_k^{-\alpha_k \tilde{s}} \left[ 1 + \alpha_k + \alpha_a \right]^{\alpha_k - \alpha_a / 2}.
\]
Noting that the LHS of (34) is negative, we get \( \tilde{s} \in (s^{SB}, \hat{s}) \), and then \( \tilde{s} = \min\{s \mid \int_{\tilde{\theta}}^\theta RdF(\theta) + \int_{\tilde{\theta}}^{\theta} \Pi_S^B(s, \theta)dF(\theta) - R - k_0 = 0\} \).

In contrast to \( \{T, z(\theta)\} \) in Prop. 4.2, Prop. 5.2 shows that after renegotiation, the bar for cooperation (\( H_1 \leq H_0 \)) is raised, and when \( H_1 \leq k_0 \leq H_0 \) the project cannot be financed. If \( L_1 < k_0 \leq H_1 \), the share given to \( V \) is always \( s^{SB} \) for \( \theta_H \geq \theta > \theta_1 \).

5.2. The case where the initial transfer can be rolled back

\( E \) has no capital at start but after entering into the contract, it is possible that \( E \) receives a transfer \( T \) from \( V \) in the first-stage investment. In fact, the transfer \( T \) can be rolled back to \( V \) in renegotiation, and the rollback would make a Pareto improvement by decreasing \( V \)'s share with a transfer from \( E \) to \( V \). Therefore, we address a more relevant contract here than that discussed in the last section.

When the participation constraint of \( V \) is not satisfied after project profitability is revealed, \( E \) can motivate \( V \) to investment in renegotiation by two ways due to the possible rollback: Giving a transfer from \( E \) to \( V \) or increasing \( V \)'s share of equity. Which way will be exploited first? Note that \( \frac{\partial\Pi^B(s, \theta)}{\partial s} < 0 \) for \( s \in [s^{SB}, \hat{s}] \), and thus raising equity results in the social welfare loss while the rollback clearly not. Hence \( E \) should choose to initiate a transfer first. Formally, we have

**Lemma 5.3.** For a given contract \( \{s, T\} \), after project profitability is revealed, if \( V \)'s participation constraint is not satisfied, \( E \) prefers to use up the transfer \( T \) first.
instead of raising V’s share until the constraint is met. Furthermore, even if V’s participation constraint is already satisfied, E had better pay a part of the transfer $T$ to V in exchange for reducing V’s share until the new share equals the best share $s^{SB}$ given by (18) or the transfer $T$ is used up.

Proof. When the participation constraint of V is not met ($\theta < \theta^{**}$), if E transfers $T_1$ ($0 \leq T_1 \leq T$) to V for motivating V to continue to invest, satisfying $\Pi_V^{SB}(s, \theta) + T_1 = R$, the net payoff of E is

$$W_1 = \Pi^{SB}(s, \theta) - \Pi_V^{SB}(s, \theta) - T_1 = \Pi^{SB}(s, \theta) - R.$$  

If E raises V’s share to $s(\theta)$ ($s(\theta) \geq s$) to attract V to invest in the second stage, meaning that $\Pi_V^{SB}(s(\theta), \theta) = R$, the payoff of E is

$$W_2 = \Pi^{SB}(s(\theta), \theta) - \Pi_V^{SB}(s(\theta), \theta) = \Pi^{SB}(s(\theta), \theta) - R.$$  

We get $W_1 > W_2$ implying that E prefers to pay the transfer first rather than raise V’s share. When the participation constraint of V is already satisfied ($\theta \geq \theta^{**}$), if E decides to transfer $T_2$ ($0 \leq T_2 \leq T$) to reduce V’s share to $s_{n2}(\theta)$ ($s^{SB} \leq s_{n2}(\theta) \leq s$) and increase the social welfare, then $\Pi_V^{SB}(s_{n2}(\theta), \theta) + T_2 = \Pi^{SB}(s, \theta)$, the net payoff of E is

$$W_3 = \Pi^{SB}(s_{n2}(\theta), \theta) - \Pi_V^{SB}(s_{n2}(\theta), \theta) - T_2 = \Pi^{SB}(s_{n2}(\theta), \theta) - \Pi^{SB}(s, \theta).$$  

If E chooses not to pay the transfer then his net payoff is

$$W_4 = \Pi^{SB}(s, \theta) - \Pi_V^{SB}(s, \theta).$$  

It is obvious that $W_3 \geq W_4$, and so E would rather pay V the transfer to decrease V’s share. 

We emphasize that as before and to be specific, we assume that E has all the bargaining power and E harvests all the negotiating surplus. We characterize the renegotiation results below.
Lemma 5.4. For a given contract \(\{s, T\}\), V’s new share \(s_{n2}(\theta)\) and the transfer \(\tilde{T}(\theta)\) from E to V are given by

\[
(s_{n2}(\theta), \tilde{T}(\theta)) = \begin{cases} 
\emptyset, & \text{if } \theta < \theta_5, \\
(s(\theta), T), & \text{if } \theta_5 \leq \theta \leq \theta_4, \\
(s, R - \Pi^{SB}_V(s, \theta)), & \text{if } \theta_4 \leq \theta \leq \theta^{**}, \\
(s_m(s, \theta), \min[\Pi^{SB}_V(s, \theta) - \Pi^{SB}_V(s^{SB}, \theta); T]), & \text{if } \theta_H \geq \theta > \theta^{**}.
\end{cases}
\]

where \(\emptyset\) indicates that the project is infeasible as before and

\[
\theta_5 = \frac{1}{A} \left( \frac{2(R - T)}{2 - \alpha_k - 2\alpha_k} \right)^{1-\alpha_k} v^{\frac{1}{2}} [u(1 - \alpha_k - \alpha_a)(1 - \hat{s})]^{-\frac{1-\alpha_k-\alpha_a}{2}} \alpha_k^{-\alpha} \hat{s}^{-\frac{1+\alpha_k+\alpha_a}{2}},
\]

\[
\theta_4 = \frac{1}{A} \left( \frac{2(R - T)}{2 - \alpha_k - 2\alpha_k} \right)^{1-\alpha_k} v^{\frac{1}{2}} [u(1 - \alpha_k - \alpha_a)(1 - s)]^{-\frac{1-\alpha_k-\alpha_a}{2}} \alpha_k^{-\alpha} (s - \frac{1 + \alpha_k + \alpha_a}{2}),
\]

\[
s(\theta) = \min\{s(\theta) | \Pi^{SB}_V(s(\theta), \theta) + T = R\},
\]

\[
s_m(s, \theta) = \max\{s_m | \Pi^{SB}_V(s_m, \theta) + \tilde{T}(\theta) = \Pi^{SB}_V(s, \theta)\}; s^{SB}].
\]

Proof. If the participation constraint of V is not met \((\theta < \theta^{**})\), E pays the transfer \(\tilde{T}\) to meet V’s participation constraint until \(T\) is used up where \(\theta_4\) is the maximum profitability threshold making all \(T\) be used up. We have \(\Pi^{SB}_V(s, \theta) + \tilde{T} = R\) for \(\theta \in [\theta_4, \theta^{**}]\). If \(T\) is not enough to make V’s participant constraint satisfied, E raises V’s shares until V’s participant constraint is satisfied. \(\theta_5\) is the minimum profitability level that makes the renegotiation successful, i.e. V abandons the project for \(\theta < \theta_5\). We have \(\Pi^{SB}_V(s(\theta), \theta) + \tilde{T} = R\) for \(\theta \in [\theta_5, \theta_4]\).

Moreover, even though V’s participation constraint is already satisfied \((\theta > \theta^{**})\), E had better pay V a fraction of the transfer \(T\) in exchange for reducing V’s share to \(s_m(s, \theta)\) until it reaches \(s^{SB}\) or it is used up. Since \(\frac{\partial \Pi^{SB}_V(s, \theta)}{\partial s} < 0\) for \(s \in [s^{SB}, s]\), the more E transfers the more the social welfare is improved. Then we get \(\Pi^{SB}_V(s_m, \theta) + \tilde{T}(\theta) = \Pi^{SB}_V(s, \theta)\), for \(\theta > \theta^{**}\). \(\square\)

In the end, the optimal contract \(\{s^{**}, T^{**}\}\) is obtained by solving the following
optimization problem:

$$\begin{align*}
\max_{s, T} \quad & \int_{\theta_5}^{\theta_4(s)} [\Pi_{E}^{SB}(s(\theta), \theta) - \tilde{T}(\theta)]dF(\theta) + \int_{\theta_5}^{\theta_{**}} [\Pi_{E}^{SB}(s, \theta) - \tilde{T}(\theta)]dF(\theta) \\
& + \int_{\theta_{**}}^{\theta_H} [\Pi_{E}^{SB}(s_m(s, \theta), \theta) - \tilde{T}(\theta)]dF(\theta) + T \}
\end{align*}$$

subject to

$$\begin{align*}
\int_{\theta_5}^{\theta_4(s)} [\Pi_{V}^{SB}(s(\theta), \theta) + \tilde{T}(\theta)]dF(\theta) + \int_{\theta_5}^{\theta_{**}} [\Pi_{V}^{SB}(s, \theta) + \tilde{T}(\theta)]dF(\theta) \\
+ \int_{\theta_{**}}^{\theta_H} [\Pi_{V}^{SB}(s_m(s, \theta), \theta) + \tilde{T}(\theta)]dF(\theta) - T & \geq R + k_0, \\
T & \geq \tilde{T}(\theta) \geq 0, \\
T & \geq 0;
\end{align*}$$

or solving

$$\begin{align*}
\max_s \quad & \int_{\theta_5}^{\theta_2} \Pi_{V}^{SB}(s(\theta), \theta)dF(\theta) + \int_{\theta_5}^{\theta_{**}} \Pi_{V}^{SB}(s, \theta)dF(\theta) \\
& + \int_{\theta_{**}}^{\theta_H} \Pi_{V}^{SB}(s_m(s, \theta), \theta)dF(\theta) - R - k_0 \}
\end{align*}$$

subject to

$$\int_{\theta_5}^{\theta_{**}} RdF(\theta) + \int_{\theta_{**}}^{\theta_H} \Pi_{V}^{SB}(s, \theta)dF(\theta) - R \geq k_0.$$

Using the method of Section 5.1, the optimal contract is summarized in the following proposition.

**Proposition 5.5.** Suppose that the transfer received by E in the first stage can be rolled back to V in renegotiation. Denote

$$L_2 \equiv \int_{\theta_5}^{\theta_2} RdF(\theta) + \int_{\theta_2}^{\theta_H} \Pi_{V}^{SB}(s^{SB}, \theta)dF(\theta) - R \geq L_1,$$

then the optimal contract \(\{s^{**}, T^*\}\) is given by (i) if \(k_0 > H_1\), the project is not feasible; (ii) if \(L_2 < k_0 \leq H_1\), \(T^* = 0\) and \(s^{**} = \bar{s} \in (s^{SB}, \tilde{s}]\), where \(\bar{s} \equiv \min\{\bar{s} | \int_{\theta_1}^{\theta_2} RdF(\theta) + \int_{\theta_1}^{\theta_H} \Pi_{V}^{SB}(\bar{s}, \theta)dF(\theta) - R - k_0 = 0\}\), and \(\bar{\theta}\) satisfies \(\Pi_{V}^{SB}(\bar{s}, \bar{\theta}) = R;\)
(iii) if $k_0 \leq L_1$, then $s^{**} = s^{SB}$, and $T^*$ is a solution of the equation

$$\int_{\theta_5}^{\theta_2} R dF(\theta) + \int_{\theta_5}^{\theta_2} \Pi^{SB}_V(s^{SB}, \theta)dF(\theta) - R - k_0 = T$$

on $T$ by noting that $\theta_5$ defined in Lemma 5.4 is a function of $T$. Moreover $T^* = L_2 - k_0$ where $L_2$ is computed with $T^*$ instead of $T$.

**Proof.** We know from Lemma 3.4 that $\max_s \Pi^{SB}_V(s, \theta) = \Pi^{SB}_V(s, \theta)$ and that

$$\Pi^{SB}_V(s, \theta) + \bar{T}(\theta) \geq R$$

if and only if $\theta \geq \theta_5$.

Therefore, V’s payoff is maximized by setting $s^{**} = \hat{s}$ and thus $\theta^{**} = \theta_1 = \theta_5$. Clearly V’s participation constraint is never satisfied if $k_0 > H_1$, where $H_1$ is defined in Prop. 5.2, i.e. the project is not feasible. If $k_0 \leq H_1$, the Lagrangian $L$ of the problem is

$$L = \int_{\theta_5}^{\theta_2} \Pi^{SB}_V(s(\theta), \theta)dF(\theta) + \int_{\theta_5}^{\theta_2} \Pi^{SB}_V(s, \theta)dF(\theta) + \int_{\theta_2}^{\theta_H} \Pi^{SB}_V(s_m(s, \theta), \theta)dF(\theta)$$

$$- R - k_0 + \gamma_2 \left( \int_{\theta_5}^{\theta_2} R dF(\theta) + \int_{\theta_2}^{\theta_H} \Pi^{SB}_V(s, \theta)dF(\theta) - R - k_0 \right),$$

where $\gamma_2$ is a non-negative Lagrange multiplier. We conclude from Kuhn-Tucher Theorem that

$$\begin{align*}
\Pi^{SB}_V(s(\theta), \theta_4(s)) &+ \int_{\theta_4(s)}^{\theta_2} \Pi^{SB}_V(s, \theta)dF(\theta) - \Pi^{SB}_V(s_m(s, \theta), \theta)ds + \int_{\theta_2}^{\theta_H} \Pi^{SB}_V(s_m(s, \theta), \theta)dF(\theta) \\
+ \int_{\theta_4(s)}^{\theta^{**}} \Pi^{SB}_V(s, \theta)dF(\theta) - \Pi^{SB}_V(s_m(s, \theta), \theta^{**})dF(\theta) + \int_{\theta_2}^{\theta_H} \frac{d\Pi^{SB}_V(s_m(s, \theta), \theta)}{ds}dF(\theta) \\
+ \gamma_2 \left( \int_{\theta_5}^{\theta_2} R dF(\theta) + \int_{\theta_2}^{\theta_H} \Pi^{SB}_V(s, \theta)dF(\theta) - R - k_0 \right) &= 0, \\
\gamma_2 &\geq 0.
\end{align*}$$

(40)

If the constraint is not binding ($k_0 \leq L_2$), we get $\gamma_2 = 0$. If $\theta = \theta_4(s)$ then $s(\theta) = s$ and if $\theta = \theta^{**}$ then $s_m(s, \theta) = s$ by the definition of $s(\theta)$ and $s_m(s, \theta)$, so we have

$$\int_{\theta_4(s)}^{\theta^{**}} \frac{d\Pi^{SB}_V(s, \theta)}{ds}dF(\theta) + \int_{\theta_2}^{\theta_H} \frac{d\Pi^{SB}_V(s_m(s, \theta), \theta)}{ds}dF(\theta) = 0.$$ 

(41)
If we set \( s = s^{SB} \) then \( s_m(s, \theta) = s^{SB} = s \), (41) holds by Prop. (3.3), indicating \( s^{**} = s^{SB} \) is a solution of the problem when the constraint is not binding. If we set \( s > s^{SB} \), the first term of (41) is negative, however, since \( s_m(s, \theta) < s \), the second term is also negative, so it contradicts with (41). We conclude that there is only one solution \( s^{SB} \), when the participation constraint is not binding. And V should give a transfer \( T^* \) to E when the agreement is signed, where \( T^* \) is a solution of the equation

\[
T = \int_{\theta_5}^{\theta_2} RdF(\theta) + \int_{\theta_2}^{\theta_H} \Pi^{SB}_V(s^{SB}, \theta)dF(\theta) - R - k_0 = L_2 - k_0 \text{ on } T.
\]

However, if the participation constraint is binding, i.e. \( H_1 \geq k_0 > L_2 \), we get \( \gamma_2 > 0 \), \( T^* = 0 \) and \( s^{**} > s^{SB} \), implying

\[
\int_{\theta_4(s)}^{\theta^{**}} d\Pi^{SB}(s, \theta) dF(\theta) + \int_{\theta^{**}}^{\theta_H} d\Pi^{SB}_V(s_m(s, \theta), \theta)dF(\theta) + \gamma_2 \int_{\theta^{**}}^{\theta_H} d\Pi^{SB}_V(s, \theta)dF(\theta) = 0.
\]

Since \( T = 0 \), we get \( \theta_4(s) = \theta^{**} \) and \( s_m(s, \theta) = s \). Similar to (34), we derive

\[
\left( \frac{d\Pi^{SB}_V(s, \theta)}{ds} \right) \left( \frac{d\Pi^{SB}_V(s, \theta)}{ds} \right)^{-1} = -\gamma_2.
\]

Substituting (14) and (21) into it, its LHS does not depend on \( \theta \). If the constraint is binding, there is a constant share \( \tilde{s} \) satisfying \( \int_{\theta_5}^{\tilde{\theta}} RdF(\tilde{\theta}) + \int_{\tilde{\theta}}^{\theta_H} \Pi^{SB}_V(\tilde{s}, \tilde{\theta})dF(\tilde{\theta}) - R - k_0 = 0 \), where \( \tilde{\theta} \) is implicitly defined by the condition \( \Pi^{SB}_V(\tilde{s}, \tilde{\theta}) = R \). We know from Prop.5.2 that \( \tilde{s} \) satisfying \( \int_{\theta_1}^{\tilde{\theta_5}} RdF(\tilde{\theta}) + \int_{\tilde{\theta}}^{\theta_H} \Pi^{SB}_V(\tilde{s}, \tilde{\theta})dF(\tilde{\theta}) - R - k_0 = 0 \), since \( \theta_5 = \theta_1 \) for \( T = 0 \), we conclude \( \tilde{s} = \tilde{s} \) and \( \tilde{\theta} = \tilde{\theta} \).

Prop. 5.5 states that if the initial investment cost \( k_0 \in [L_2, H_1] \), the initial transfer \( T^* = 0 \), and naturally, the optimal contract is the same as that given by Prop. 5.2. If the rollback is permitted, it naturally brings new benefits of investing in the project, which are totally harvested by V following our model assumption. Particularly, it increases the investment threshold from \( \theta_1 \) to \( \theta_5 \), and in this way, it increases \( T^* \) as well, which further enlarges the advantage of the rollback.

To make a comparison between the long-term contingent contract given by Prop. 4.2 with the contract considering renegotiation, if the initial sunk cost \( k_0 \in (L_1, H_1) \), we have \( T^* = 0 \), and thus it suffices to compare Prop. 4.2 with Prop. 5.2. The result \( H_0 > H_1 \) indicates long-term one-off contracts create more possibilities for
cooperation. Whether renegotiable contracts improve V’s share may depend on the relationship between two exogenous variables $\theta_L$ and $R$. We define a critical value $R_1$ satisfying

$$\int_{\theta_1}^{\theta} R_1 dF(\theta) = \int_{\theta_L}^{\theta} \Pi_V^{SB}(s, \theta)dF(\theta).$$

If $R = R_1$ then $\tilde{s} = \bar{s}$. If $R > R_1$, we have $\tilde{s} > \bar{s}$, indicating that the renegotiation increases V’s payoff.

On the other hand, if the initial sunk cost is sufficient low or the project is profitable enough, V’s participant constraint is met, and moreover we have $T > 0$. Thus it suffices to compare Prop. 4.2 with Prop. 5.5. We define another critical value $R_2$ satisfying

$$\int_{\theta_1}^{\theta} R_2 dF(\theta) = \int_{\theta_L}^{\theta} \Pi_V^{SB}(s, \theta)dF(\theta).$$

If $R = R_2$ then $L_2 = L_0$, the two contracts have the same unconstrained cut-off point. If $R > R_2$, we get $L_2 > L_0$ implying V will give more transfer in the negotiable setting for the same initial investment $k_0$. If $R < R_2$, we get the opposite conclusion.

**Figure 5.** Optimal contract and renegotiation results versus project profitability under the case of (a) $T^* = 0$ and of (b) $T^* > 0$, where $a_a=0.4$, $\alpha_k=0.2$, $\theta_L=3.2$, $\theta_H=5.2$, $u = v = A=1$, $R = 0.5$, $\theta \sim U[\theta_L, \theta_H]$. 

Fig. 5 depicts the optimal contract and renegotiation results versus project profitability under two cases: One is the initial investment cost satisfying $L_1 = L_2 < k_0 \leq H_1$, i.e. the project is feasible but not very profitable, leading to $T^* = 0$, and
the other is \( k_0 \leq L_2 \), indicating that the project is sufficient profitable relative to the initial sunk cost and \( T^* > 0 \). As (5.1) predicted, Fig. 5a says that if \( \theta < \theta_1 \) the project is abandoned, if \( \theta > \bar{\theta} \) there is no renegotiation, and otherwise V’s share of equity is raised up to his participation constraint is satisfied. Fig. 5b tells the same story in addition to that the rollback from E to V is exploited to meet V’s participation constraint.

\[(a) \quad T^* = 0 \text{ if } k_0 = 0.5 \text{ and } k_0 \in [L_2, H_1] \quad (b) \quad T^* > 0 \text{ if } k_0 = 0.3 \text{ and } k_0 \leq L_2\]

Figure 6. The optimal total social welfare versus project profitability under the case of (a) \( T^* = 0 \) and of (b) \( T^* > 0 \), where \( \alpha_a=0.4, \alpha_k=0.2, \theta_L=3.2, \theta_H=5.2, \) \( u = v = A=1, \) \( R = 0.5, \) \( \theta \sim U[\theta_L, \theta_H]. \)

Fig. 6 presents the total social welfare instead of the contract versus the profitability, and the other conditions are the same with Fig. 5. It says clearly that how renegotiation improves social welfare. It is noteworthy that after \( \theta \) is disclosed, as long as V’s participation constraint is satisfied, no renegotiation occurs.

6. Conclusion

Mainly due to information asymmetry, startups are difficult to raise outside capital for investing in a project. Venture capital is popular to solve the problem over the world where a venture capitalist (V) provides the entrepreneur (E) with precious advisor services as well as capital. Such financing mode is very helpful to reduce information asymmetry and have acquired a great success in practice. However, it incurs double-sided moral hazard since the project output depends on both E and V’s effort which are their private information. Moreover, the investment scale has
a considerable effect on project success. To make a successful cooperation between E and V, it is extremely important to effectively split equity between them and the equity split is the unique incentive factor. In this paper, we borrow the well-known Cobb-Douglas output function of V’s investment scale, E’s and V’s effort to develop a two-stage investment model. To be specific, we assume that E has no money to invest and V must pay all investment costs. V works in a fully competitive venture capital market, and thus E has all the bargaining power. We use a backward recursion method to investigate the optimal contract with and without renegotiation for the two-stage investment.

Our analysis shows that if V’s participation constraint is not binding, the shares of equity in equilibrium allocated to participants only depend on their output elasticities in both single-stage and two-stage financing. If V’s participation constraint is binding, the optimal V’s share decreases with profitability prospects in one-stage financing, while in two-stage financing it remains constant, regardless of whether the information on profitability prospects is verifiable or not. In two-stage financing, if the contract is not renegotiable, V’s share is a constant for all continuation states, and V definitely invests during the expansion period.

Our findings indicate that renegotiation occurs only when V’s participation constraint is not met after a project profitability is revealed. In the two-stage investment, the initial investment cost, or equivalently project prospects, has an important impact on the contract design. Only when the cost is too large or the total project profitability on average is not high enough, the project is infeasible.

Our study provides new findings in venture capital financing. First, if project prospects are good enough, we get an explicit optimal equity split, which is totally determined by the inputs’ output elasticities. Second, in contrast to single-stage financing, if project profitability is contractible, we get an explicit constant equity split and a possible cash transfer from V to E before the profitability is revealed. Third, if the contract can be renegotiable in the second stage, once the potential profitability is disclosed, we fix the thresholds determining whether E should abandon the project, whether E should go ahead without doing anything, or should increase V’s equity or transfer cash to V until V’s participation constraint is satisfied.
Our analysis provides some inspiration for further studies. First, the syndication of venture capital investments is a well-established phenomenon in the market, as documented in Lerner (1994). Venture capitalists typically feel more assured with an arrangement when other venture capitalists with comparable proficiency are prepared to invest. In a typical syndication arrangement, there is one lead investor who takes charge of the project’s financing, followed by several other venture capitalists who also invest in the project. Second, exploring the double-sided moral hazard problems when the investment is made in stages without knowing the distribution of potential profitability (ambiguity). One would establish a model with a foreseen cost according to incomplete contract theory. Third, one can reallocate the bargaining power of the participants. Moreover, when E knows project’s quality but V doesn’t, the adverse selection problem arises in the venture capital market. It can be inferred that V would provide E with a menu of contracts to choose to invest in a project with appropriate quality. Last, another subject is to analyse the double-sided moral hazard problem in a continuous-time model. We leave them for future research.

References


