Automatica 141 (2022) 110310

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Gradient-based bearing-only formation control: An elevation angle approach ☆

Liangming Chen [a,*], Zhiyong Sun [b]

[a] School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore
[b] Department of Electrical Engineering, Eindhoven University of Technology, The Netherlands

ARTICLE INFO

Article history:
Received 1 March 2021
Received in revised form 2 January 2022
Accepted 17 March 2022
Available online 13 April 2022

Keywords:
Elevation angle rigidity
Multi-agent systems
Formation control
Bearing measurement
Gradient controller

ABSTRACT

This paper proposes an elevation angle rigidity theory in both 2D and 3D spaces, and applies it to solve multi-agent formation control with only inter-agent bearing/direction measurements in agents' local coordinate frames. Motivated by the sensor technology in measuring elevation angle and angular diameter, we develop elevation angle rigidity by attaching each agent in a multi-agent framework with a rod in 2D and a ball in 3D, respectively. By defining the elevation angle rigidity matrix, conditions for infinitesimal elevation angle rigidity are derived. Compared to previously developed angle rigidity-based and bearing rigidity-based formation control laws, the proposed elevation angle rigidity-based control law can maintain the gradient-based control property. Compared to distance-based formation control laws, less sensor measurements are required. The formation maneuvering with desired translation and rotation is also realized by using only local bearing measurements. Simulation examples illustrate the advantages and effectiveness of the proposed approach.

1. Introduction

Motivated by the applications in mechanical structures, biological materials and multi-agent networks (Thorpe & Duxbury, 1999), distance rigidity has been a popular and powerful tool in analyzing stability and structures of distance-constrained frameworks since the 1970s (Asimow & Roth, 1978). By using a set of distance constraints on the corresponding edges of a graph, all the distances in a rigid framework are maintained when its vertices' positions are perturbed locally (Anderson, Yu, Fidan, & Hendrickx, 2008). By describing a geometric shape by a set of distance constraints, distance rigidity has been successfully utilized as a key tool in formation shape control of multiple autonomous agents (Anderson et al., 2008; Han, Lin, & Fu, 2015; Olfati-Saber & Murray, 2002; Yang, Cao, Fang, & Chen, 2018). Most of the formation control laws developed upon distance rigidity are the gradient of a potential function consisting of distance errors (Krick, Broucke, & Francis, 2009). Therefore, each agent's measurements in distance rigidity-based formation control laws are mainly inter-agent relative positions (Dimarogonas & Johansson, 2009). Note that these relative positions can be measured in agents' local coordinate frames since distance constraints remain the same under different coordinate frames (Ahn, 2020).

Recently, many other types of rigidity concepts are developed, including bearing rigidity (Zhao & Zelazo, 2016), angle rigidity (Chen, Cao, & Li, 2021; Jing, Zhang, Lee, & Wang, 2019), ratio-of-distance rigidity (Cao, Han, Li, & Xie, 2019) and their mixture with distance rigidity (Kwon, Sun, Anderson, & Ahn, 2019). These novel rigidity theories introduce some new constraints to guarantee the uniqueness of multi-agent frameworks, e.g., bearings, angles and their mixture with distances. From the application perspective, these rigidity notions also enable novel design of formation control laws that involve different sensor measurements to achieve a desired multi-agent formation. For example, based on the developed bearing rigidity, bearing-only formation control law is proposed in Zhao and Zelazo (2016) to almost globally stabilize a target formation. Since an inter-agent bearing can be measured by monocular cameras, sonars and passive radars, bearing-only formation control laws have less stringent requirement on agents' sensor measurement (Zhao & Zelazo, 2019) than the distance rigidity-based formation laws. However, bearing measurement is coordinate-dependent, which implies that the alignment of all agents' local coordinate frames with that of a global coordinate frame is required in implementing the proposed bearing-only formation control law. Different from bearing constraints, each interior angle constraint formed by three agents is coordinate-free, which motivates the study of angle rigidity (Chen et al., 2021; Jing et al., 2019). By choosing the

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Bert Tanner under the direction of Editor Christos G. Cassandras.

* Corresponding author.
E-mail addresses: liangming.chen@ntu.edu.sg (L. Chen), z.sun@tue.nl (Z. Sun).

https://doi.org/10.1016/j.automatica.2022.110310
0005-1098/

cosine of angle constraints as the angle function, a gradient-based control law is proposed in Jing et al. (2019) which requires local relative position measurements and inter-agent communication of neighbors' real-time angle errors. In Chen et al. (2021), signed angle constraints are defined and the designed control law only uses local bearing/direction measurements. The desired angle-constrained formation shape in Chen et al. (2021) is constructed through a sequential and directed form, while there does not exist a corresponding potential function whose gradient corresponds to the designed control law in Chen et al. (2021). However the gradient-based control structure is a favored property in the local and the global stability analysis of a general formation.

Recently, the paper Chan, Jayawardhana, and de Marina (2021) has first proposed to attach each planar agent with a circular disk to solve the bearing-only formation control problem, in which a gradient-based formation control law is designed using the information of the measured bearings. Motivated by recent advances of sensor technology in measuring elevation angle and angular diameter, we propose a novel solution in this paper to solve the bearing-only formation problem in both 2D and 3D spaces. First, we develop an elevation angle rigidity theory by attaching each agent in the multi-agent framework with a rod in 2D and a ball in 3D, under which the elevation angle and angular diameter can be measured by available sensing technologies, e.g., cameras or sensor arrays. Then, we define the cotangent and cosecant of the elevation angle as the elevation angle function in 2D and 3D, respectively. By defining the elevation angle errors as the potential function, we propose a gradient-based bearing-only control law for stabilizing a target formation. The advantages of this solution include that the bearing-only control law can be implemented locally without using the information of the rod's height or ball's radius, that the desired formation can be more general than the existing angle-constrained sequential formations, and that the gradient control property is maintained.

The rest of this paper is organized as follows. Section 2 gives the definition of elevation angle and its rigidity in 2D. Section 3 introduces infinitesimal elevation angle rigidity. The application to gradient-based bearing-only formation control is given in Section 4. Then, Section 5 presents the elevation angle rigidity and its application in 3D. Simulations are provided in Section 6.

2. Elevation angle and its rigidity in 2D

In this section, we introduce the concept of elevation angle and define elevation angle rigidity in 2D.

2.1. Elevation angle

An undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ consists of a node set $\mathcal{V} = \{1,\ldots,N\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ with the number of elements $M = |\mathcal{E}|$, in which $(i,j)\in\mathcal{E} \Leftrightarrow (j,i)\in\mathcal{E}$. The set of neighbors of node i is denoted as $\mathcal{N}_i = \{j\in\mathcal{V} | (i,j)\in\mathcal{E}\}$. A framework $\mathcal{F}(\mathcal{G},p)$ is a combination of an undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ and an embedding p with $p = [p_1^{\mathrm{T}},\ldots,p_N^{\mathrm{T}}]^{\mathrm{T}}$, $p_i\in\mathbb{R}^3$, $\forall i = 1,\ldots,N$. When we consider the 2D case, all points are in the XOY plane, i.e., $p_i = [x_i, y_i, z_i]^{\mathrm{T}}$ and $z_i = 0, \forall i\in\mathcal{V}$, under which we call $\mathcal{F}(\mathcal{G},p)$ a planar framework, and p a planar configuration. We assume that there are no overlapping points in p, i.e., $p_i \neq p_j$, $i\neq j$. The bearing $b_{ij}\in\mathbb{R}^3$ is defined as a unit vector starting from p_i and pointing towards p_j ($p_i\neq p_j$), i.e.,

$$b_{ij} = \frac{p_j - p_i}{\|p_j - p_i\|} = \frac{p_{ij}}{l_{ij}} \tag{1}$$

To develop the elevation angle rigidity, we consider that each node i is attached with a vertical rod with height $h_i > 0$. Then, the position of the end point i' of the rod upon node i can be

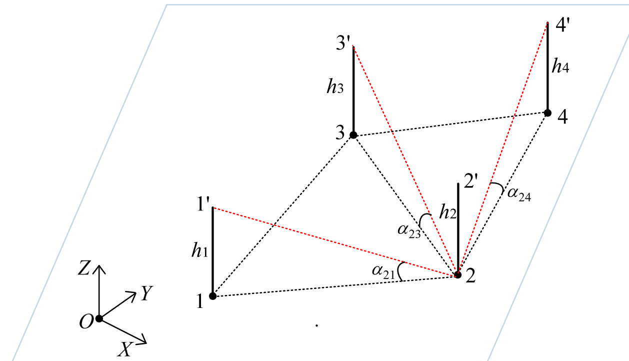


Fig. 1. Three elevation angles $\measuredangle i2i'$, $i = 1,3,4$ in node 2.

calculated by $p_{i'} = p_i + [0,0,h_i]^{\mathrm{T}}$. Now, we define the elevation angle $\alpha_{ij}\in(0,\frac{\pi}{2})$ from node i towards node j as

$$\alpha_{ij} := \measuredangle jij' = \arccos(b_{ij}^{\mathrm{T}} b_{ij'}) = \arctan\left(h_j/l_{ij}\right) \tag{2}$$

If all nodes' rods are with the same height $h_i = h_j = h_c$, $\forall i\in\mathcal{V}$, then it follows that

$$\alpha_{ij} = \alpha_{ji} = \arctan\left(h_c/l_{ij}\right) \tag{3}$$

Correspondingly, $\alpha_{ij}\neq\alpha_{ji}$ if $h_i\neq h_j$. In this work, we consider the case that rods' heights $h_i = h_c > 0$ are the same for all nodes $\forall i\in\mathcal{V}$. As shown in Fig. 1, three elevation angles $\alpha_{21},\alpha_{23},\alpha_{24}$ are defined for node 2.

2.2. Elevation angle rigidity

Before giving the definition for elevation angle rigidity, we first give the definitions of equivalence and congruence between two frameworks.

Definition 1. Two frameworks $\mathcal{F}_1(\mathcal{G},p)$ and $\mathcal{F}_2(\mathcal{G},q)$ with the same graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ are *equivalent* if

$$\alpha_{ij}(p_i,p_j) = \alpha_{ij}(q_i,q_j),\ \forall (i,j)\in\mathcal{E}. \tag{4}$$

They are *congruent* if

$$\alpha_{ij}(p_i,p_j) = \alpha_{ij}(q_i,q_j),\ \forall i,j\in\mathcal{V}. \tag{5}$$

According to the definitions of equivalent and congruent frameworks, we now define elevation angle rigidity and global elevation angle rigidity in 2D.

Definition 2 (*Elevation Angle Rigidity*)**.** A planar framework $\mathcal{F}(\mathcal{G},p)$ is *elevation angle rigid* if there exists an $\epsilon > 0$ such that every planar framework $\mathcal{F}'(\mathcal{G},q)$ that is equivalent to it and satisfies $\|q-p\| < \epsilon$, is congruent to it.

Definition 3 (*Global Elevation Angle Rigidity*)**.** A planar framework $\mathcal{F}(\mathcal{G},p)$ is *globally elevation angle rigid* if every planar framework that is equivalent to it is also congruent to it.

3. Infinitesimal elevation angle rigidity in 2D

Similar to distance rigidity, elevation angle rigidity matrix plays an important role in evaluating infinitesimal elevation angle rigidity. We first introduce the elevation angle function, whose partial derivative with respect to the position vector defines elevation angle rigidity matrix.

For a given planar framework $\mathcal{F}(\mathcal{G},p)$, we define the *elevation angle function* $f_E(p):\mathbb{R}^{3N}\to\mathbb{R}^M$ by

$$f_E(p) := [f_1,\ldots,f_M]^{\top}, \tag{6}$$

where

$$f_k := \cot\alpha_k = \frac{\cos\alpha_{ij}}{\sin\alpha_{ij}} = \frac{l_{ij}}{h_c},\ k = 1,\ldots,M \tag{7}$$

is the mapping from the position vector $[p_i^\top, p_j^\top]^\top$ of the kth edge (i,j) in $\mathcal{E}$ to the cotangent of the elevation angle α_{ij}. Note that $\alpha_k := \alpha_{ij} = \alpha_{ji}$ corresponds to the elevation angle associated with the kth edge (i,j) that connects nodes i and j. Since $\alpha_{ij} \in (0, \frac{\pi}{2})$, one has $f_k \in (0,\infty)$. Using this elevation angle function, one can define elevation angle rigidity matrix.

3.1. Elevation angle rigidity matrix

We consider the kth edge (i,j) in $\mathcal{E}$ and taking the time-derivative of f_k yields

$$\frac{\mathrm{d}f_k}{\mathrm{d}t} = h_c^{-1}\frac{\mathrm{d}l_{ij}}{\mathrm{d}t} = h_c^{-1}b_{ij}^\top(\dot{p}_j - \dot{p}_i) = \frac{\partial f_k}{\partial p_i}\dot{p}_i + \frac{\partial f_k}{\partial p_j}\dot{p}_j \tag{8}$$

which implies $\frac{\partial f_k}{\partial p_i} = -h_c^{-1}b_{ij}^\top \in \mathbb{R}^{1\times 3}$ and $\frac{\partial f_k}{\partial p_j} = h_c^{-1}b_{ij}^\top \in \mathbb{R}^{1\times 3}$. Based on this fact, we define an elevation angle rigidity matrix by taking the partial derivative of elevation angle function $f_E(p)$ with respect to the position vector p

$$\frac{\mathrm{d}f_E(p)}{\mathrm{d}t} = \frac{\partial f_E(p)}{\partial p}\dot{p} = R_e(p)\dot{p}, \tag{9}$$

where $R_e(p) \in \mathbb{R}^{M\times 3N}$ is called the *elevation angle rigidity matrix*, whose rows are indexed by the elements of $\mathcal{E}$ and columns are the vertices, i.e.,

$$R_e(p) = \frac{\partial f_E(p)}{\partial p} =$$

$$\begin{array}{c} \\ \text{Edge } 1 \\ \cdots \\ \text{Edge } k \text{ of } (i,j) \\ \cdots \\ \text{Edge } M \end{array} \begin{array}{c} \begin{array}{ccccc} \cdots & \text{Vertex } i & \cdots & \text{Vertex } j & \cdots \end{array} \\ \left[\begin{array}{ccccc} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -h_c^{-1}b_{ij}^\top & 0 & h_c^{-1}b_{ij}^\top & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right] \end{array} \tag{10}$$

Note that the nonzero element in the distance rigidity matrix is the transpose of relative position p_{ij}, while in the elevation angle rigidity matrix, the corresponding element is the transpose of bearing b_{ij}. Since the framework is in the XOY plane, the Z-axis component in b_{ij} will be zero, which yields a zero column under each vertex i in (10).

For a planar framework $\mathcal{F}(\mathcal{G},p)$, its elevation angle-preserving motions satisfy $\dot{f}_E(p) = R_e(p)\dot{p} = 0$ which include translation and rotation of the framework $\mathcal{F}(\mathcal{G},p)$ in the XOY plane. Therefore, the null space of the elevation angle rigidity matrix $R_e(p)$ includes translation and rotation with respect to the framework, which can be described by linear subspaces spanned by the three vectors $\xi_1 = \mathbf{1}_N \otimes [1,0,0]^\top$, $\xi_2 = \mathbf{1}_N \otimes [0,1,0]^\top$, $\xi_3 = \left[(Q_0p_1)^\top,\ (Q_0p_2)^\top,\ \cdots,\ (Q_0p_N)^\top\right]^\top$, where $Q_0 = \left[\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ is a skew-symmetric matrix, $\otimes$ represents Kronecker product and $\mathbf{1}_N$ denotes the $N\times 1$ column vector of all ones. Note that ξ_1 and ξ_2 correspond to translation motions along the X and Y axes, respectively, and ξ_3 corresponds to a rotation of the framework in the XOY plane. Then, one has the following lemma.

Lemma 1 (*Rank of Elevation Angle Rigidity Matrix*)**.** *For an elevation angle rigidity matrix $R_e(p)$, it always holds that Span$\{\xi_1,\xi_2,\xi_3\} \subseteq$ Null$(R_e(p))$ and Rank$(R_e(p)) \le 2N-3$.*

Proof. Because each row sum of $R_e(p)$ equals zero, one has $R_e(p)\xi_1 = 0$ and $R_e(p)\xi_2 = 0$. Taking an arbitrary row in $R_e(p)$ as an example, one has the corresponding row in $R_e(p)\xi_3$

$$-h_c^{-1}b_{ij}^\top Q_0 p_i + h_c^{-1}b_{ij}^\top Q_0 p_j = h_c^{-1}l_{ij}b_{ij}^\top Q_0 b_{ij} = 0 \tag{11}$$

where we have used the fact that $b_{ij}^\top Q_0 b_{ij} = 0$ holds for skew-symmetric matrix Q_0. Therefore, one has Span$\{\xi_1,\xi_2,\xi_3\} \subseteq$ Null$(R_e(p))$. It is obvious that ξ_1,ξ_2 are linearly independent. Suppose that ξ_3 is linearly dependent to ξ_1,ξ_2, then $\xi_3 = a\xi_1 + b\xi_2$ with at least one nonzero $a \in \mathbb{R}$ or $b \in \mathbb{R}$, which implies $p_i = p_j, \forall i,j \in \mathcal{V}$. Because there are no overlapping points in p, one has that ξ_1,ξ_2,ξ_3 are linearly independent.

Since there are N columns in $R_e(p)$ whose elements are zero and at least three linearly independent vectors ξ_1,ξ_2,ξ_3 in the null space of $R_e(p)$, one has Rank$(R_e(p)) = 3N -$ Null$(R_e(p)) \le 3N-(N+3) = 2N-3$. □

3.2. Infinitesimal elevation angle rigidity

First, we introduce infinitesimal elevation angle rigid motion. Consider each $p_i, \forall i \in \mathcal{V}$ of $\mathcal{F}(\mathcal{G},p)$ is on a differentiable path. Define the path $p(t)$ as an *infinitesimal elevation angle rigid motion* of $\mathcal{F}$ if on the path $f_E(p)$ keeps constant, i.e., $\dot{f}_E(p) = R_e(p)\dot{p} \equiv 0$. We say such an infinitesimal elevation angle rigid motion $p(t)$ is *trivial* if it can be given by

$$p_i(t) = \mathcal{Q}(t)p_i(t_0) + \mathcal{W}(t), \forall i \in \mathcal{V}, t \ge 0, \tag{12}$$

where $\mathcal{Q}(t) \in \mathbb{R}^{3\times 3}$ and $\mathcal{W}(t) \in \mathbb{R}^3$ describe the rotation matrix and the translation of the framework in the XOY plane, respectively, and $\mathcal{Q}(t)$, $\mathcal{W}(t)$ are all differentiable functions.

Definition 4 (*Infinitesimal Elevation Angle Rigidity*)**.** A planar framework $\mathcal{F}(\mathcal{G},p)$ is *infinitesimally elevation angle rigid* if all its infinitesimal elevation angle rigid motions $p(t)$ are trivial.

Now, we present the necessary and sufficient condition to check infinitesimal elevation angle rigidity.

Theorem 1. *A planar framework $\mathcal{F}(\mathcal{G},p)$ is infinitesimally elevation angle rigid if and only if the rank of its elevation angle rigidity matrix $R_e(p)$ is $2N-3$.*

Now we give the relationship between elevation angle rigidity and infinitesimal elevation angle rigidity.

Theorem 2. *If a planar framework $\mathcal{F}(\mathcal{G},p)$ is infinitesimally elevation angle rigid, then it is elevation angle rigid.*

The proof of Theorems 1–2 is straightforward by following Chen et al. (2021, Theorems 4, 6).

3.3. Relationship between elevation angle rigidity and distance rigidity

Both distance rigidity and elevation angle rigidity are defined upon the framework $\mathcal{F}(\mathcal{G},p)$. The following proposition gives the relationship between elevation angle rigidity and distance rigidity for planar configurations.

Proposition 1. *For a given planar framework $\mathcal{F}(\mathcal{G},p)$, it is distance rigid if and only if it is elevation angle rigid.*

Proof. According to the definition of elevation angle in (2), one has that for the case $h_i = h_j = h_c$, $\alpha_{ij}(p_i,p_j) = \alpha_{ij}(q_i,q_j)$ if and only if $l_{ij}(p_i,p_j) = l_{ij}(q_i,q_j)$. Therefore, according to the definition of distance rigidity, one concludes that they are equivalent. □

For infinitesimal rigidity, we also have the corresponding proposition.

Proposition 2. *For a given planar framework $\mathcal{F}(\mathcal{G},p)$, it is infinitesimally distance rigid if and only if it is infinitesimally elevation angle rigid.*

Proof. Note that the distance rigidity matrix can be described by

$$R_d(p) = \frac{\partial f_D(p)}{\partial p} = h_c \text{diag}\{l_{ij}\} R_e(p) \tag{13}$$

where $f_D(p) = [\cdots, \|p_i - p_j\|^2, \dots]^{\mathrm{T}}$, $\text{diag}\{l_{ij}\} \in \mathbb{R}^{M \times M}$ is a diagonal matrix with the distance of the kth edge as its diagonal element. Because there are no overlapping points in position vector p, the diagonal matrix $\text{diag}\{l_{ij}\}$ is a nonsingular matrix with full rank. Therefore, one has that

$$\text{Rank}(R_d(p)) = \text{Rank}(R_e(p)) \tag{14}$$

According to Theorem 1, one has that the rank equality $\text{Rank}(R_e(p)) = 2N - 3$ is a necessary and sufficient condition for infinitesimal elevation angle rigidity. Therefore, one concludes the equivalence between infinitesimal distance rigidity and infinitesimal elevation angle rigidity. □

When p is generic (Connelly, Jordán, & Whiteley, 2013) and the framework $\mathcal{F}(\mathcal{G}, p)$ is infinitesimally elevation angle rigid, with a slight abuse of notions, we also say the graph $\mathcal{G}$ is infinitesimally elevation angle rigid. When $\mathcal{F}(\mathcal{G}, p)$ is infinitesimally elevation angle rigid and $|\mathcal{E}| = 2N - 3$, we say $\mathcal{F}(\mathcal{G}, p)$ is infinitesimally and minimally elevation angle rigid.

Remark 1. If agents have different h_i, then $\alpha_{ij} \neq \alpha_{ji}$. In this case, $l_{ij} = l_{ji} \neq h_i f_k$ because $f_{(i,j)} \neq f_{(j,i)}$ and f_k depends on the direction between i and j. This reveals the difference between elevation angle function and distance rigidity function. For an undirected graph, one always has $l_{ij} = l_{ji}$ in distance rigidity function. However, this is not the case for elevation angle function since $f_{(i,j)} \neq f_{(j,i)}$ when h_i are different, which gives more design freedom and properties than that in the distance rigidity function.

4. Bearing-only formation control in 2D

In this section, we employ the developed elevation angle rigidity in 2D to the application of bearing-only formation control. We propose a novel bearing-only formation control law by taking the gradient of a properly defined potential function. The objective is to design control law by using bearing-only information under an infinitesimally elevation angle rigid graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ such that

$$\lim_{t\to\infty} e_{ij}(t) = \lim_{t\to\infty} (\alpha_{ij}(t) - \alpha_{ij}^*) = 0 \tag{15}$$

where $\alpha_{ij}^* \in (0, \frac{\pi}{2})$ is the desired elevation angle from agent i to agent j.

4.1. Gradient-based formation stabilization

For an agent i moving in the XOY plane, we consider its dynamics governed by

$$\dot{p}_i = u_i, i = 1, \dots, N, \tag{16}$$

where $p_i = [x_i, y_i, z_i]^{\mathrm{T}} \in \mathbb{R}^3$ denotes agent i's position, $u_i = [u_{xi}, u_{yi}, u_{zi}]^{\mathrm{T}} \in \mathbb{R}^3$ is the control input to be designed, and N is the number of agents in the group. Since all agents lie in XOY plane, one always has $z_i = 0$ and $u_{zi} = 0$. The constraint is that agent i can only measure bearings $b_{ij}, b_{ij'}$ with respect to its neighboring agent $j, j \in \mathcal{N}_i$.

In angle rigidity-based formation control (Jing et al., 2019), the gradient of a potential function consisting of angle errors is not a bearing-only control law. Conversely, there does not exist a corresponding potential function for the proposed angle rigidity-based bearing-only control law (Chen et al., 2021). To obtain both favorable properties, we now design an elevation angle rigidity-based bearing-only control law which is the gradient of a potential function. To be specific, we design the potential function as

$$\begin{aligned} V_s &= \sum_{i=1}^{N} V_{si} = \frac{h_c}{4} \sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} (\cot\alpha_{ij} - \cot\alpha_{ij}^*)^2 \\ &= \frac{h_c}{2} (f_E(p) - f_E^*)^{\mathrm{T}} (f_E(p) - f_E^*) \end{aligned} \tag{17}$$

where $f_E^* = [\cot\alpha_1^*, \dots, \cot\alpha_k^*, \dots, \cot\alpha_M^*]^{\mathrm{T}}$. Taking the gradient of (17) with respect to p_i yields the control law for agent i

$$\begin{aligned} u_i &= -(\frac{\partial V_s}{\partial p_i})^{\mathrm{T}} \\ &= -h_c \sum_{j\in\mathcal{N}_i} (\frac{\partial \cot\alpha_{ij}}{\partial p_i})^{\mathrm{T}} (\cot\alpha_{ij} - \cot\alpha_{ij}^*) \end{aligned} \tag{18}$$

Combining (8) with (18), one has the control law as

$$\begin{aligned} u_i &= \sum_{j\in\mathcal{N}_i} b_{ij} (\cot\alpha_{ij} - \cot\alpha_{ij}^*) \\ &= \sum_{j\in\mathcal{N}_i} b_{ij} \left(\frac{b_{ij}^{\mathrm{T}} b_{ij'}}{\sqrt{1 - (b_{ij}^{\mathrm{T}} b_{ij'})^2}} - \cot\alpha_{ij}^* \right) \end{aligned} \tag{19}$$

in which only bearings b_{ij} and $b_{ij'}$ are used in the final form of (19). Note that the information of rod's height h_c is not required in control law (19). Before giving the main result, we introduce some important lemmas.

Lemma 2. *If a planar framework $\mathcal{F}(\mathcal{G}, p)$ is infinitesimally and minimally elevation angle rigid, then $R_e(p) R_e^{\mathrm{T}}(p)$ is positive definite.*

Proof. When $\mathcal{F}(\mathcal{G}, p)$ is infinitesimally minimally elevation angle rigid, according to Lemma 1, one has $\text{Rank}(R_e(p)) = 2N - 3$ and $\text{Null}(R_e(p)) = N + 3$. Therefore, $R_e^{\mathrm{T}}(p) R_e(p) \in \mathbb{R}^{3N\times 3N}$ has $(N + 3)$ zero eigenvalues. Because $R_e^{\mathrm{T}}(p) R_e(p)$ is positive semi-definite, the other $(2N - 3)$ eigenvalues of $R_e^{\mathrm{T}}(p) R_e(p)$ are all positive. Note that $R_e(p) R_e^{\mathrm{T}}(p) \in \mathbb{R}^{(2N-3)\times(2N-3)}$ shares the same $(2N - 3)$ eigenvalues as $R_e^{\mathrm{T}}(p) R_e(p)$, but $R_e^{\mathrm{T}}(p) R_e(p)$ has extra $3N - (2N - 3) = N + 3$ zero eigenvalues. Then, it follows that all the eigenvalues of $R_e(p) R_e^{\mathrm{T}}(p)$ are all positive, i.e., $R_e(p) R_e^{\mathrm{T}}(p)$ is positive definite. □

Lemma 3. *For implementing the bearing-only control law (19), each agent can use its own local coordinate frame to measure the bearings b_{ij} and $b_{ij'}$, and a global coordinate frame is not required.*

The proof of Lemma 3 follows similarly to that in distance-based formation control law (e.g., Lemma 2 of Sun, Garcia de Marina, Anderson, & Cao, 2018) and is omitted here. Therefore, different from the bearing-only formation control law (Zhao & Zelazo, 2016), the proposed bearing-only control law (19) does not require a common or global coordinate frame for bearing measurements and control implementation.

Now, we present the first main result on bearing-only formation stabilization based on elevation angle rigidity.

Theorem 3. *Consider an N-agent system governed by (16) in the XOY plane. If the graph $\mathcal{G}$ is infinitesimally and minimally elevation angle rigid, under the control law (19), all agents will locally and asymptotically achieve the desired elevation angles defined in (15).*

Proof. We first write all agents' dynamics into a compact form

$$\begin{aligned} \dot{p} = u &= -\left(\frac{\partial V_s}{\partial p}\right)^{\mathrm{T}} = -\left(\frac{\partial V_s}{\partial f_E(p)} \frac{\partial f_E(p)}{\partial p}\right)^{\mathrm{T}} \\ &= -\left[h_c (f_E(p) - f_E^*)^{\mathrm{T}} R_e(p)\right]^{\mathrm{T}} \\ &= -h_c R_e^{\mathrm{T}}(p) \left(f_E(p) - f_E^*\right) \end{aligned} \tag{20}$$

Then, it follows that

$$\dot{V}_s = \frac{\partial V_s}{\partial p}\dot{p} = -\left(\frac{\partial V_s}{\partial p}\right)\left(\frac{\partial V_s}{\partial p}\right)^{\mathrm{T}}$$
$$= -h_c^2(f_E(p)-f_E^*)^{\mathrm{T}}R_e(p)R_e^{\mathrm{T}}(p)\left(f_E(p)-f_E^*\right)$$

According to Lemma 2, $R_e(p)R_e^{\mathrm{T}}(p)$ is positive definite when the trajectory of p lies in a neighborhood set $\mathbb{U}_1$ of the desired equilibrium $\{p \in \mathbb{R}^{3N} | f_E(p) = f_E^*\}$. Note that $\mathbb{U}_1$ exists since $\dot{V}_s \leq 0$. Therefore, for $p \in \mathbb{U}_1$, one has $\dot{V}_s < 0$ and

$$\dot{V}_s \leq -h_c^2 \lambda_{\min}\|f_E(p)-f_E^*\|^2 \tag{21}$$

where $\lambda_{\min} = \min\{\lambda_{\min}\left(R_e(p)R_e^{\mathrm{T}}(p)\right), \forall p \in \mathbb{U}_1\}$. Then, one has the local exponential convergence of $\cot\alpha_{ij} - \cot\alpha_{ij}^*$, which implies the asymptotic convergence of $e_{ij} = \alpha_{ij} - \alpha_{ij}^*$. □

Remark 2. Compared to Dimarogonas and Johansson (2009), the proof in Theorem 3 constructs a special form of potential function V_s whose gradient (19) only requires local bearing measurements, in contrast to the relative position measurements needed in Dimarogonas and Johansson (2009). Note that gradient controller is a favorable property for the formation control system, while several popular bearing-only formation controllers (Zhao & Zelazo, 2016) are not gradient-based. Instead of focusing on tree or cyclic formations, uniquely determined elevation angle rigid formations are achieved in Theorem 3, where the formation error $\|f_E(p) - f_E^*\|$ converges to zero exponentially.

Now, we conduct the collision avoidance analysis to obtain the bound on the inter-agent initial distances, under which the control law (19) is well-defined during the evolution. Note that

$$\begin{aligned} l_{ij}(t) =& l_{ij}(0) + \int_0^t \dot{l}_{ij}(\tau)d\tau = l_{ij}(0) + \int_0^t b_{ij}^{\mathrm{T}}(\dot{p}_j - \dot{p}_i)d\tau \\ =& l_{ij}(0) + \int_0^t b_{ij}^{\mathrm{T}}\Big(\sum_{k\in\mathcal{N}_j} b_{jk}(\cot\alpha_{jk} - \cot\alpha_{jk}^*) \\ &- \sum_{j\in\mathcal{N}_i} b_{ij}(\cot\alpha_{ij} - \cot\alpha_{ij}^*)\Big)d\tau \\ \geq& l_{ij}(0) - \int_0^t 2\sum_{i=1}^{M} |\cot\alpha_i - \cot\alpha_i^*| d\tau \end{aligned} \tag{22}$$

Using the fact that $\|x\|_1 \leq \sqrt{n}\|x\|_2$ for $x \in \mathbb{R}^n$, one has $\sum_{i=1}^M |\cot\alpha_i - \cot\alpha_i^*| \leq \sqrt{M}\|f_E(p) - f_E^*\|$. Then it follows that $l_{ij}(t) \geq l_{ij}(0) - 2\sqrt{M}\int_0^t \|f_E(p) - f_E^*\| d\tau$. According to (21), one has

$$\dot{V}_s \leq -2h_c\lambda_{\min}V_s \tag{23}$$

which implies $V_s(t) \leq V(0)e^{-2h_c\lambda_{\min}t}$. Therefore, one has

$$\|f_E(p) - f_E^*\| = \sqrt{2V_s/h_c} \leq \sqrt{2V_s(0)/h_c}e^{-h_c\lambda_{\min}t} \tag{24}$$

Hence, one has

$$l_{ij}(t) \geq l_{ij}(0) - \frac{2}{h_c\lambda_{\min}}\sqrt{\frac{2MV_s(0)}{h_c}}(1 - e^{-h_c\lambda_{\min}t}) \tag{25}$$

Finally, we come to the following conclusion.

Theorem 4. *Consider an N-agent system governed by (16) in the XOY plane is controlled by (19) and the graph $\mathcal{G}$ is infinitesimally and minimally elevation angle rigid. There exists a positive constant κ such that if $l_{ij}(0) > \frac{2}{h_c\kappa}\sqrt{\frac{2MV_s(0)}{h_c}}$, $\forall(i,j) \in \mathcal{E}$, then all the neighboring agents will not collide with each other for $\forall t > 0$.*

Proof. Since $l_{ij}(0) > 0$, $\exists T_1 > 0$ such that in $[0, T_1)$, no collision happens between agents i and j. Assume that collision may occur between agents i and j in $[T_1, \infty)$, then there must exist a time instant T_c such that $l_{ij}(T_c) = 0$. Since no collision happens in $[T_1, T_c^-)$, the closed-loop system is well-defined in $[T_1, T_c^-)$. Letting $\kappa = \lambda_{min}$ and following the calculations in (22)–(25), one has that $l_{ij}(T_c^-) \geq l_{ij}(0) - \frac{2}{h_c\lambda_{\min}}\sqrt{\frac{2MV_s(0)}{h_c}} > 0$ which is bounded away from zero. This contradicts the assumption that collision happens at T_c. Thus, no collision will occur in $[0, \infty)$. □

Table 1
Comparison between distance-based and elevation angle-based approaches.

Property	Approach: Distance-based	Elevation angle-based
Shape specifications	Distances l_{ij}^*	Elevation angles α_{ij}^*
Sensor measurements	Relative positions $p_i - p_j$	Bearings $b_{ij}, b_{ij'}$

Remark 3. By setting $h_c\cot\alpha_{ij}^* = l_{ij}^*$, the potential function (17) can also be seen as a realization of the potential function for distance-based formation control (Dimarogonas & Johansson, 2009; Sun, Mou, Anderson, & Cao, 2016). However, in terms of the formation shape specification and required measurements, the controller (19) is quite different from the controllers in Dimarogonas and Johansson (2009) and Sun et al. (2016). Under the same potential function, our gradient-based elevation angle formation controller only needs local bearing measurements, while the gradient-based distance rigidity formation controller needs relative position measurements. This justifies the advantages of the elevation angle approach. Table 1 summarizes the difference between distance-based formation approach and elevation angle-based formation approach.

Remark 4. Based on the measurements b_{ij} and $b_{ij'}$, agent i can derive the scaled distance l_{ij}/h_c with a common h_c. Under this fact, compared to the distance formation approach, the advantages of using the elevation angle approach lie in two aspects. The first is that the control law (19) is independent of h_c. The second is that h_i can be different from h_j, which offers more flexibility for distributed formation systems.

4.2. Formation maneuvering

To steer the formation maneuver with desired translation, rotation and a desired size, we design a bearing-only formation maneuvering control law as

$$\begin{aligned} u_i &= \sum_{j\in\mathcal{N}_i} b_{ij}(\cot\alpha_{ij} - \gamma_c\cot\alpha_{ij}^* + \mu_{ij}) \\ &= \sum_{j\in\mathcal{N}_i} b_{ij}(\cot\alpha_{ij} - \gamma_c\cot\alpha_{ij}^*) + \sum_{j\in\mathcal{N}_i}\mu_{ij}b_{ij} \end{aligned} \tag{26}$$

where $\gamma_c \in \mathbb{R}^+$ is used to adjust the size of the formation, $\mu_{ij} \in \mathbb{R}$ is used to realize the desired collective translation and rotation.

When $\gamma_c \neq 1$ and $\mu_{ij} = 0, (i,j) \in \mathcal{E}$, we prove that the desired formation size can be achieved. By using (7), one has $\cot\alpha_{ij} - \gamma_c\cot\alpha_{ij}^* = \frac{l_{ij} - r_c l_{ij}^*}{h_c}$ where l_{ij}^* is the distance corresponding to the desired angle α_{ij}^*. By following the proof in Theorem 3, one has that $\lim_{t\to\infty}(\cot\alpha_{ij}(t) - \gamma_c\cot\alpha_{ij}^*) = 0$ which implies that $\lim_{t\to\infty}(l_{ij}(t) - \gamma_c l_{ij}^*) = 0$. Therefore, by adjusting the parameter γ_c in all the agents, the formation size characterized by the distance $\gamma_c l_{ij}^*$ between agents can be achieved. Specifically, $0 < \gamma_c < 1$ corresponds to formation shrinking and $\gamma_c > 1$ corresponds to formation enlargement. Since γ_c is a constant number, all agents can get it by low-cost communication.

When $\gamma_c \neq 0$ and $\mu_{ij} \neq 0, (i,j) \in \mathcal{E}$, we illustrate how to design μ_{ij} such that translational and rotational maneuvering can be realized. Note that when the desired formation shape is achieved, one has $\lim_{t\to\infty}\dot{p}_i(t) = \lim_{t\to\infty}u_i(t) = \lim_{t\to\infty}\sum_{j\in\mathcal{N}_i}\mu_{ij}b_{ij}(t)$. Therefore, we can properly design μ_{ij} such that $\sum_{j\in\mathcal{N}_i}\mu_{ij}b_{ij}$ forms the desired translational velocity and rotational velocity. Let $v_c^* = [v_{cx}^*, v_{cy}^*, 0]^{\mathrm{T}}$ be the translational velocity in the XOY plane, and $\omega_c^* \in \mathbb{R}$ be the rotational speed. Consider a reference configuration $p^* = [p_1^{*\top}, \ldots, p_N^{*\top}]^\top$ which satisfies all the desired elevation angles α_{ij}^*. Then, we define bearing $b_{ij}^* = \frac{p_j^*-p_i^*}{\|p_j^*-p_i^*\|}$ and formation centroid $p_c^* = \frac{\sum_{i=1}^N p_i^*}{N}$. Now, we design the parameters μ_{ij} by

$$\sum_{j\in\mathcal{N}_i}\mu_{ij}b_{ij}^* = v_c^* + \omega_c^* Q_0(p_c^* - p_i^*) \tag{27}$$

Writing (37) into a compact form yields

$$h_c R^{\mathrm{T}}(p^*)\mu = 1_N \otimes v_c^* + \omega_c^*(Q_0 \otimes I_N)(1_N \otimes p_c^* - p^*) \tag{28}$$

where $\mu = [\mu_1, \ldots, \mu_M]^{\mathrm{T}} \otimes 1_3$. Then, one can calculate

$$\mu = h_c^{-1}\left(R^{\mathrm{T}}(p^*)\right)^{\dagger}\left[1_N \otimes v_c^* + \omega_c^*(Q_0 \otimes I_N)(1_N \otimes p_c^* - p^*)\right]$$

where $\left(R^{\mathrm{T}}(p^*)\right)^{\dagger}$ denotes the pseudo-inverse of matrix $R^{\mathrm{T}}(p^*)$. Now, we have the following main result on bearing-only formation maneuvering.

Theorem 5. *Consider an N-agent system governed by (16) in the XOY plane. If the graph $\mathcal{G}$ is infinitesimally and minimally elevation angle rigid and μ_{ij} are sufficiently small, under the maneuvering control law (26) with (28), all agents will locally achieve the desired formation with desired scale described by γ_c, and desired motion with translation speed $\|v_c^*\|$ and rotation speed ω_c^*.*

Proof. First, the compact form of all agents' dynamics is derived as

$$\dot{p} = -h_c R_e^{\mathrm{T}}(p)\left(f_E(p) - \gamma_c f_E^*\right) + h_c R_e^{\mathrm{T}}(p^*)\mu \tag{29}$$

Taking the time-derivative of (17) yields

$$\begin{aligned}\dot{V}_s =& \frac{\partial V_s}{\partial p}\dot{p} = -\left(\frac{\partial V_s}{\partial p}\right)\left(\frac{\partial V_s}{\partial p}\right)^{\mathrm{T}} \\ =& -h_c^2(f_E(p) - \gamma_c f_E^*)^{\mathrm{T}} R_e(p) R_e^{\mathrm{T}}(p)\left(f_E(p) - \gamma_c f_E^*\right) \\ &+ h_c^2(f_E(p) - \gamma_c f_E^*)^{\mathrm{T}} R_e(p) R_e^{\mathrm{T}}(p^*)\mu \\ \le& -h_c^2(\lambda_{\min} - \beta)\|f_E(p) - \gamma_c f_E^*\|^2\end{aligned} \tag{30}$$

where we consider the states confined in the set $\mathbb{S} = \{p \in \mathbb{R}^{3N} | \|f_E(p) - \gamma_c f_E^*\| \le \rho\}$, where $R_e(p)R_e^{\mathrm{T}}(p^*)\mu$ is locally Lipschitz with respect to $f_E(p) - \gamma_c f_E^*$, $R_e(p^*)R_e^{\mathrm{T}}(p^*)\mu = 0$, $\lambda_{\min}$ is the minimum positive eigenvalue of $R_e(p)R_e^{\mathrm{T}}(p)$ in $\mathbb{S}$, and β and ρ are positive constants. Since μ_{ij} and β are sufficiently small, one has $\lim_{t\to\infty}\left(f_E(p(t)) - \gamma_c f_E^*\right) = 0$. Using (26), one has

$$\begin{aligned}\lim_{t\to\infty}\dot{p}_i(t) &= \lim_{t\to\infty}\sum_{j\in\mathcal{N}_i}\mu_{ij}b_{ij}(t) \\ &= \lim_{t\to\infty}R_*^{\mathrm{T}}\left(v_c^* + \omega_c^* Q_0(p_c^* - p_i^*)\right)\end{aligned} \tag{31}$$

where $R_*^{\mathrm{T}} \in \mathbb{R}^{3\times 3}$ describes the rotation from b_{ij}^* to $b_{ij}(t)$ along the Z-axis. Since $\|R_*^{\mathrm{T}}v_c^*\| = \|v_c^*\|$ and $R_*^{\mathrm{T}}Q_0(p_c^* - p_i^*) = R_*^{\mathrm{T}}Q_0R_t^*(p_c(t) - p_i(t)) = Q_0(p_c(t) - p_i(t))$, (31) implies that the desired translation speed $\|v_c^*\|$ and rotation speed ω_c^* are achieved. □

5. Extension to 3D case

In this section, we extend the elevation angle rigidity to 3D case where all agents lie in $\mathbb{R}^3$. Since some parts in 3D case are similar to 2D case, we only introduce the differences.

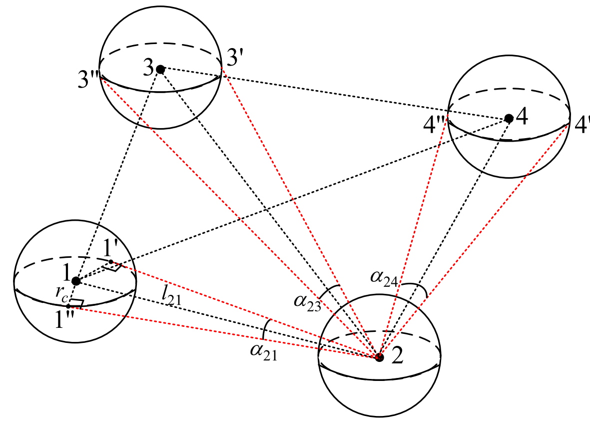


Fig. 2. Three elevation angles $\measuredangle j'2j''$, $j = 1, 3, 4$ in node 2 in 3D.

5.1. Elevation angle in 3D

In 3D case, we suppose that each node is auxiliarily attached with a (physical or virtual) ball with radius $r_c > 0$. The elevation angle $\bar{\alpha}_{ij} \in (0, \pi/3)$ from point i to point j in 3D is defined as

$$\bar{\alpha}_{ij} := \measuredangle j'ij'' = \arccos(b_{ij'}^{\mathrm{T}}b_{ij''}) = 2\measuredangle j'ij = 2\arcsin\frac{r_c}{l_{ij}}$$

where j' and j'' are the two points in agent j's ball, j, j', j'', i are coplanar and $b_{ij'}^{\mathrm{T}}b_{jj'} = 0$, $b_{ij''}^{\mathrm{T}}b_{jj''} = 0$. When $l_{ij} = 2r_c$, i.e., agent i's ball touches agent j's ball in its surfaces, one has $\bar{\alpha}_{ij} = 2\arcsin\frac{r_c}{2r_c} = \frac{\pi}{3}$. Hence, the range of $\bar{\alpha}_{ij}$ belongs to $(0, \pi/3)$ if i's ball does not collide with agent j's ball (see Fig. 2).

Now, we introduce the elevation angle function in 3D. For each framework $\mathcal{F}(\mathcal{G}, p)$, we define the *elevation angle function* $\bar{f}_E(p) : \mathbb{R}^{3N} \to \mathbb{R}^M$ by

$$\bar{f}_E(p) := [\bar{f}_1, \ldots, \bar{f}_M]^T,$$

where

$$\bar{f}_k := \csc\frac{\bar{\alpha}_k}{2} = \csc\frac{\bar{\alpha}_{ij}}{2} = \frac{1}{\sin\frac{\bar{\alpha}_{ij}}{2}} = \frac{l_{ij}}{r_c}, \ k = 1, \ldots, M \tag{32}$$

is the mapping from the position vector $[p_i^{\mathrm{T}}, p_j^{\mathrm{T}}]^{\mathrm{T}}$ of the kth edge (i, j) in $\mathcal{E}$ to the cosecant of the half of elevation angle $\bar{\alpha}_{ij}$. Since $\bar{\alpha}_{ij} \in (0, \frac{\pi}{3})$, $\bar{f}_k \in (2r_c, \infty)$. Using this elevation angle function, one can define elevation angle rigidity and infinitesimal elevation angle rigidity in 3D by following Sections 2 and 3, and details are omitted here.

5.2. Gradient-based formation stabilization

To develop a gradient-based formation control law, we first design the potential function as

$$\begin{aligned}\bar{V}_s &= \sum_{i=1}^N \bar{V}_{si} = \frac{r_c}{4}\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}(\csc\bar{\alpha}_{ij} - \csc\bar{\alpha}_{ij}^*)^2 \\ &= \frac{r_c}{2}(\bar{f}_E(p) - \bar{f}_E^*)^{\mathrm{T}}(\bar{f}_E(p) - \bar{f}_E^*)\end{aligned} \tag{33}$$

where $\bar{f}_E^* = [\cot\frac{\bar{\alpha}_1^*}{2}, \ldots, \cot\frac{\bar{\alpha}_k^*}{2}, \ldots, \cot\frac{\bar{\alpha}_M^*}{2}]^{\mathrm{T}}$. Taking the gradient of (33) with respect to p_i yields the control law for agent i

$$\begin{aligned}u_i &= -\left(\frac{\partial \bar{V}_s}{\partial p_i}\right)^{\mathrm{T}} \\ &= -r_c\sum_{j\in\mathcal{N}_i}\left(\frac{\partial\csc\bar{\alpha}_{ij}}{\partial p_i}\right)^{\mathrm{T}}(\csc\frac{\bar{\alpha}_{ij}}{2} - \csc\frac{\bar{\alpha}_{ij}^*}{2})\end{aligned} \tag{34}$$

Substituting (32) into (34), the control law can be written as

$$u_i = \sum_{j\in\mathcal{N}_i} b_{ij}(\csc\frac{\bar{\alpha}_{ij}}{2} - \csc\frac{\bar{\alpha}^*_{ij}}{2})$$
$$= \sum_{j\in\mathcal{N}_i} b_{ij}\left(\frac{1}{\sqrt{(1-b^{\mathrm{T}}_{ij'}b_{ij''})/2}} - \csc\frac{\bar{\alpha}^*_{ij}}{2}\right) \quad (35)$$

where only bearing information of b_{ij} and $b_{ij'}, b_{ij''}$ is required, and we have used the fact that $\sin^2(\frac{\bar{\alpha}_{ij}}{2}) = \frac{1-\cos\bar{\alpha}_{ij}}{2}$ and $\sin\frac{\bar{\alpha}_{ij}}{2} > 0$. Note that the information of the ball's radius r_c is not required in control law (35). With the gradient-based control law (35), we present the main result.

Proposition 3. *Consider an N-agent system governed by (16) in 3D. If the graph $\mathcal{G}$ is infinitesimally and minimally elevation angle rigid, under the control law (35), all agents will locally and asymptotically achieve the desired elevation angles.*

Proof. The proof is straightforward by following Lemma 2 and Theorem 3. □

5.3. Formation maneuvering

To achieve collective motions, we design the bearing-only formation maneuvering law as

$$u_i = \sum_{j\in\mathcal{N}_i} b_{ij}(\csc\frac{\bar{\alpha}_{ij}}{2} - \gamma_c\csc\frac{\bar{\alpha}^*_{ij}}{2} + \mu_{ij}) \quad (36)$$
$$= \sum_{j\in\mathcal{N}_i} b_{ij}\left(\frac{1}{\sqrt{(1-b^{\mathrm{T}}_{ij'}b_{ij''})/2}} - \gamma_c\csc\frac{\bar{\alpha}^*_{ij}}{2}\right) + \sum_{j\in\mathcal{N}_i}\mu_{ij}b_{ij}$$

Similarly, by using the reference formation configuration p^*, the parameters μ_{ij} satisfy

$$\sum_{j\in\mathcal{N}_i}\mu_{ij}b^*_{ij} = v^*_c + \bar{\omega}^*_c \times (p^*_c - p^*_i) \quad (37)$$

where $\times$ denotes the cross product operation and $\bar{\omega}^*_c \in \mathbb{R}^3$ is the desired angular velocity vector of the elevation angle rigid formation. Writing (37) into a compact form yields

$$\mu = r_c^{-1}\left(R^{\mathrm{T}}(p^*)\right)^{\dagger}[1_N \otimes v^*_c + (\bar{\omega}^*_c \otimes I_N) \times (1_N \otimes p^*_c - p^*)]$$

Following (29)–(31), one can obtain similar results about formation maneuvering in 3D.

Remark 5. Compared with (Chan et al., 2021), the control laws (19) and (35) are structurally simpler and also do not need the information of the rod's height and the ball's radius. Furthermore, the required bearing measurements in control laws (19) and (35) can be obtained from vision-based target locating (Pachter, Ceccarelli, & Chandler, 2007) or sensor arrays (Nielsen, 1994) in agents' local coordinate frames, which facilitates distributed implementation of angle-constrained formations.

Remark 6. Note that the 2D elevation angle can also be defined by following the way of defining 3D elevation angle or attaching each node with a disk as employed in Chan et al. (2021). However, due to the physical restrictions such as minimum inter-agent distance to meet collision avoidance and visibility constraint, we choose the way of attaching each node with a rod to define the 2D elevation angle, which is more practical to facilitate bearing measurements.

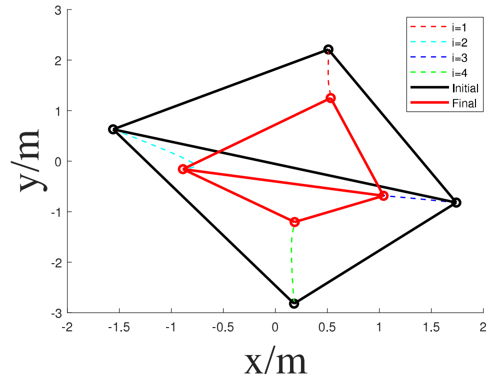


Fig. 3. Trajectories in 2D formation stabilization.

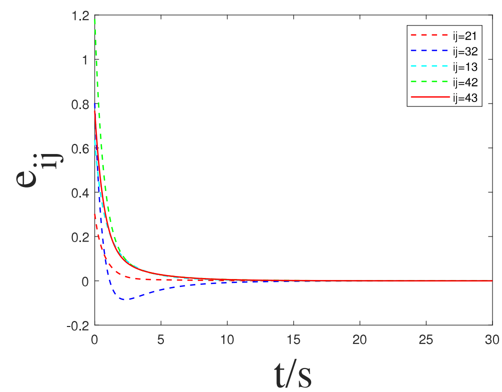


Fig. 4. Elevation angle errors in 2D formation stabilization.

6. Simulation

In this section, we use simulation examples in 2D and 3D to show the effectiveness and advantages of the proposed bearing-only formation control laws.

We consider a four-agent formation graph with five edges where $h_c = r_c = 2$. The initial states in the 2D case are set as: $p_1(0) = [0.51; 2.21; 0]$, $p_2(0) = [-1.56; 0.63; 0]$, $p_3(0) = [1.74; -0.82; 0]$, $p_4(0) = [0.18; -2.82; 0]$. The desired elevation angles are: $\alpha^*_{12} = \alpha^*_{21} = \pi/4, \alpha^*_{13} = \alpha^*_{31} = \pi/4, \alpha^*_{32} = \alpha^*_{23} = \pi/4$, $\alpha^*_{42} = \alpha^*_{24} = 0.93$, $\alpha^*_{43} = \alpha^*_{34} = 1.11$. For the 3D case, the parameter changes are $p_4(0) = [0.18; -2.82; 5.21]$, $\alpha^*_{43} = \alpha^*_{34} = 0.80$, $\alpha^*_{41} = \alpha^*_{14} = 0.65$.

Figs. 3 and 4 show the agents' trajectories and the change of elevation angle errors, respectively, in the task of 2D formation stabilization. Figs. 5 and 6 present the agents' trajectories and the change of elevation angle errors in 2D, respectively, in the task of 2D formation maneuvering when passing through narrow passages with obstacle avoidance. Figs. 7–8 present the agents' trajectories and the change of elevation angle errors in 3D, respectively. It can be seen from Figs. 4, 6, and 8 that the elevation angle errors converge to zero in the three tasks.

7. Conclusion

With the motivation of proposing a scalable bearing-only formation control law, this paper has developed elevation angle rigidity in both 2D and 3D cases. By attaching each node with a rod in 2D and a ball in 3D, a desired formation has been determined by a set of elevation angle constraints. Then, the infinitesimal elevation angle rigidity has been developed by studying a proposed elevation angle rigidity matrix. We have further

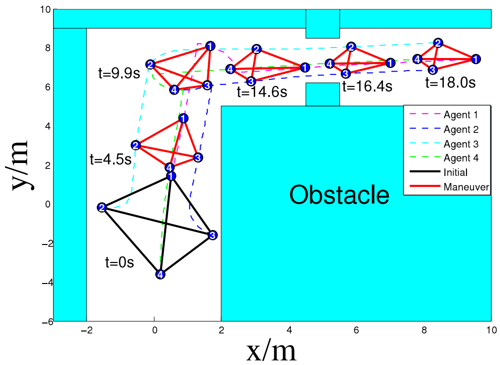


Fig. 5. Trajectories in 2D formation maneuvering.

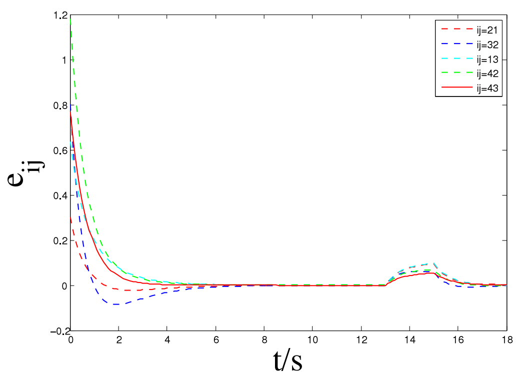


Fig. 6. Elevation angle errors in 2D formation maneuvering.

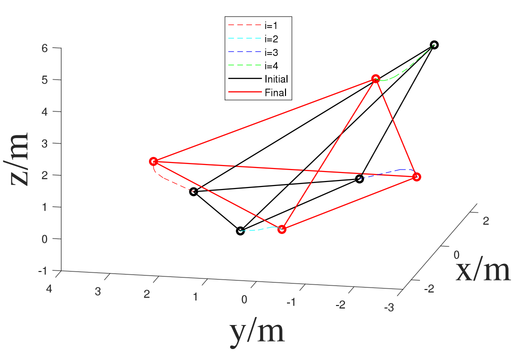


Fig. 7. Trajectories in 3D formation stabilization.

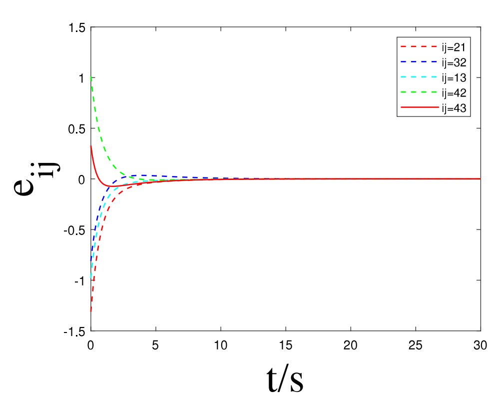


Fig. 8. Elevation angle errors in 3D formation stabilization.

proposed bearing-only formation stabilization law to achieve the desired formation, and bearing-only formation maneuvering law to achieve the desired translation and rotation motions, respectively. The main advantages of the proposed elevation angle-based formation control law are that it has a gradient property and only local bearing measurements are needed. Future work will concentrate on designing globally or almost globally stable bearing-only multi-agent formation control law based on the advantage of the gradient property.

References

Ahn, Hyo-Sung (2020). *Formation control*. Springer.

Anderson, Brian D. O., Yu, Changbin, Fidan, Baris, & Hendrickx, Julien M. (2008). Rigid graph control architectures for autonomous formations. *IEEE Control Systems Magazine*, *28*(6), 48–63.

Asimow, Leonard, & Roth, Ben (1978). The rigidity of graphs. *Transactions of the American Mathematical Society*, *245*(11), 279–289.

Cao, Kun, Han, Zhimin, Li, Xiuxian, & Xie, Lihua (2019). Ratio-of-distance rigidity theory with application to similar formation control. *IEEE Transactions on Automatic Control*, *65*(6), 2598–2611.

Chan, N. P. K., Jayawardhana, B., & de Marina, H. G. (2021). Angle-constrained formation control for circular mobile robots. *IEEE Control Systems Letters*, *5*(1), 109–114.

Chen, Liangming, Cao, Ming, & Li, Chuanjiang (2021). Angle rigidity and its usage to stabilize multi-agent formations in 2D. *IEEE Transactions on Automatic Control*, *66*(8), 3667–3681.

Connelly, Robert, Jordán, Tibor, & Whiteley, Walter (2013). Generic global rigidity of body–bar frameworks. *Journal of Combinatorial Theory, Series B*, *103*(6), 689–705.

Dimarogonas, Dimos V., & Johansson, Karl H. (2009). Further results on the stability of distance-based multi-robot formations. In *2009 American control conference* (pp. 2972–2977). IEEE.

Han, Tingrui, Lin, Zhiyun, & Fu, Minyue (2015). Three-dimensional formation merging control under directed and switching topologies. *Automatica*, *58*(8), 99–105.

Jing, Gangshan, Zhang, Guofeng, Lee, Heung Wing Joseph, & Wang, Long (2019). Angle-based shape determination theory of planar graphs with application to formation stabilization. *Automatica*, *105*(7), 117–129.

Krick, Laura, Broucke, Mireille E., & Francis, Bruce A. (2009). Stabilisation of infinitesimally rigid formations of multi-robot networks. *International Journal of Control*, *82*(3), 423–439.

Kwon, Seong-Ho, Sun, Zhiyong, Anderson, Brian D. O., & Ahn, Hyo-Sung (2019). Hybrid distance-angle rigidity theory with signed constraints and its applications to formation shape control. arXiv preprint arXiv:1912.12952.

Nielsen, Richard O. (1994). Azimuth and elevation angle estimation with a three-dimensional array. *IEEE Journal of Oceanic Engineering*, *19*(1), 84–86.

Olfati-Saber, Reza, & Murray, Richard M. (2002). Graph rigidity and distributed formation stabilization of multi-vehicle systems. In *Proceedings of the 41st IEEE conference on decision and control, 2002: Vol. 3* (pp. 2965–2971). IEEE.

Pachter, Meir, Ceccarelli, Nicola, & Chandler, Philip R. (2007). Vision-based target geo-location using camera equipped mavs. In *Proceedings of the 46th IEEE conference on decision and control* (pp. 2333–2338). IEEE.

Sun, Zhiyong, Garcia de Marina, Hector, Anderson, Brian D. O., & Cao, Ming (2018). Quantization effects and convergence properties of rigid formation control systems with quantized distance measurements. *International Journal of Robust and Nonlinear Control*, *28*(16), 4865–4884.

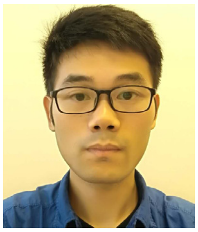
Sun, Zhiyong, Mou, Shaoshuai, Anderson, Brian D. O., & Cao, Ming (2016). Exponential stability for formation control systems with generalized controllers: A unified approach. *Systems & Control Letters*, *93*(7), 50–57.

Thorpe, Michael F., & Duxbury, Phillip M. (1999). *Rigidity theory and applications*. Springer Science & Business Media.

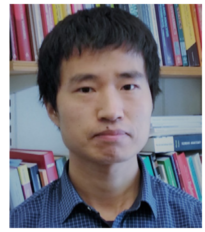
Yang, Qingkai, Cao, Ming, Fang, Hao, & Chen, Jie (2018). Constructing universally rigid tensegrity frameworks with application in multiagent formation control. *IEEE Transactions on Automatic Control*, *64*(1), 381–388.

Zhao, Shiyu, & Zelazo, Daniel (2016). Bearing rigidity and almost global bearing-only formation stabilization. *IEEE Transactions on Automatic Control*, *61*(5), 1255–1268.

Zhao, Shiyu, & Zelazo, Daniel (2019). Bearing rigidity theory and its applications for control and estimation of network systems: Life beyond distance rigidity. *IEEE Control Systems Magazine*, *39*(2), 66–83.



Liangming Chen received the B.E. degree in Automation from Southwest Jiaotong University, China, in 2015. From 2015 to 2021, he has enrolled jointly in the Ph.D. programs of systems and control in Harbin Institute of Technology, China and University of Groningen, the Netherlands. Currently, he is a research fellow in Nanyang Technological University, Singapore. His current research interests are in rigidity theory, formation control, and multi-agent systems.



Zhiyong Sun received the Ph.D. degree from The Australian National University (ANU), Canberra ACT, Australia, in February 2017. He was a Research Fellow/Lecturer with the Research School of Engineering, ANU, from 2017 to 2018. From June 2018 to January 2020, he worked as a postdoctoral researcher at Department of Automatic Control, Lund University, Lund, Sweden. Since January 2020 he has joined Eindhoven University of Technology (TU/e), the Netherlands, as an assistant professor. His research interests include multi-robotic systems, control of autonomous formations, distributed control and optimization.