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Triangular angle rigidity for distributed localization in 2D☆

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A R T I C L E I N F O

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Recent advances in sensing technology have enabled sensor nodes to measure interior angles with respect to their neighboring nodes. However, it is unknown which combination of angle measurements is necessary to make a sensor network localizable, and it is also unidentified if there is a distributed localization algorithm whose required communication only consists of the sensor nodes' measured angles and estimated positions. Motivated by these two challenging problems, this paper develops *triangular angle rigidity* for those networks consisting of a set of nodes and triangular angle constraints in 2D. First, we transfer the geometric constraint of each triangle into an angle-induced linear constraint. Based on the linear constraint, we show that different from angle rigidity, triangular angle rigidity implies global triangular angle rigidity. More importantly, inspired by Laman's theorem, we propose a topological, necessary and sufficient condition to check generic triangular angle rigidity. Based on the results on triangular angle rigidity, both algebraic and topological localizability conditions are developed, which are necessary and sufficient when the number of anchor nodes in the network is two. Both continuous and discrete localization algorithms are proposed, in which only measured angles and estimated positions are communicated among the sensor nodes. Finally, a simulation example with 32 sensor nodes is used to validate the effectiveness of the proposed approaches.

1. Introduction

Sensor network localization has been extensively studied due to its wide applications in, e.g., robotics (Nguyen, Qiu, Nguyen, Cao, & Xie, 2019). The aim of network localization is to determine the positions of free nodes using their sensor measurements with respect to their neighbors and communication information from their neighbors (Aspnes et al., 2006; Mao, Fidan, & Anderson, 2007). Three kinds of measurements are often employed in the localization of sensor networks, namely, relative positions, distances, and bearings. When relative position measurements are available, distributed localization algorithms have been developed in Lin, Fu, and Diao (2015). Since an inter-node relative position consists of distance and bearing information, the sensor measurement cost will be reduced if only one of them is required for localization. This prompts the development of distributed localization approaches using distance-only measurements (Diao, Lin, & Fu, 2014; Han, Lin, Zheng, Han, & Zhang, 2017; Jiang, Anderson, & Hmam, 2019). Using the localized positions from distance measurements, formation control algorithms have been designed to achieve a desired multi-agent formation (Cao, Yu, & Anderson, 2011; Jiang, Deghat, & Anderson, 2016; Nguyen et al., 2019). Bearing-only network localization has also received growing interest since bearing sensing is passive, low-cost and light-weight (Zhao & Zelazo, 2019). Two types of bearing sensing have been studied, namely bearing sensing with and without the alignment of all sensor nodes' coordinate frames. For the first type, bearing-only network localization has been studied with focuses on rigidity-based localizability conditions (Eren, Whiteley, & Belhumeur, 2006; Zhao & Zelazo, 2016b), noisy bearing measurements (Shames, Bishop, & Anderson, 2012; Ye, Anderson, & Yu, 2017), and coordinate frame alignment by orientation estimation (Li, Luo, & Zhao, 2019; Trinh, Lee, Ye, & Ahn, 2018), just name a few. For the second type, localization algorithms have also been proposed, where the sensor nodes' coordinate frames can be chosen with arbitrary orientations (Cao, Han, Lin, & Xie, 2021; Lin, Han, Zheng, & Fu, 2016).

In addition to these three kinds of measurements, interior angle measurements within triangles have been becoming accessible, particularly from the angle of arrival and angle of departure modules embedded in the latest Bluetooth 5.1 technology (Cominelli, Patras, & Gringoli, 2019). Thus, it is crucial to identify angle-only localizability conditions and develop angle-only localization algorithms, which, however, have not been adequately investigated. To solve these two problems, we choose to use the tool of rigidity theory since distance rigidity and bearing rigidity have been used to solve the localization of distance-constrained networks (Eren et al., 2004; Jiang et al., 2019) and

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bearing-constrained networks (Eren et al., 2006; Zhao & Zelazo, 2016b), respectively. Thus, to solve the triangular angle-constrained network localization problem, we propose *triangular angle rigidity*, which describes the property that under the given triangular angle constraints, the network can only translate, rotate or scale when its points are perturbed locally. Based on the developed results on triangular angle rigidity, the two challenging problems, namely angle-only localizability and angle-only localization, are solved.

The paper's contributions are summarized as follows.

(1) Triangular angle rigidity: Different from angle rigidity (Chen, Cao, & Li, 2021), triangular angle rigidity implies global triangular angle rigidity. More importantly, we propose a topological, necessary and sufficient condition to check generic triangular angle rigidity.

(2) Localizability conditions: A new angle-induced linear constraint is proposed to identify localizability of triangular angle-constrained networks. Different from those sufficiently topological localizability conditions (Cao et al., 2021; Jing, Wan, & Dai, 2022; Lin et al., 2016) and algebraic localizability conditions (Fang, Li, & Xie, 2020; Jing et al., 2022), both the proposed algebraic and topological localizability conditions are necessary and sufficient when the number of anchor nodes is two.

(3) Localization algorithms: Different from those localization algorithms whose required communication includes measured bearing vectors (Cao et al., 2021; Jing et al., 2022; Lin et al., 2016; Zhao & Zelazo, 2016b), our localization algorithms require the communication of interior angles which are scalars and totally independent of sensor nodes' coordinate frames. Compared with the 2D network localization using distance measurements where each sensor node is required to have at least three neighbors (Diao et al., 2014), each sensor node in our localization algorithms is allowed to have only two neighbors.

The rest of the paper is organized as follows. Section 2 presents the preliminaries. Section 3 discusses the triangular angle rigidity. Section 4 presents the localizability conditions. Section 5 proposes localization algorithms. Simulation examples are presented in Section 6.

2. Preliminaries

2.1. Notations

Consider a 2D static sensor network consisting of n_a anchor nodes and n_f free nodes. Let $\mathcal{V}_a = \{1, 2, \ldots, n_a\}$ be the set of anchor nodes, whose positions, denoted by $p_a = [p_1^\top, \ldots, p_{n_a}^\top]^\top \in \mathbb{R}^{2n_a}$, are known by themselves. Let $\mathcal{V}_f = \{n_a+1, \ldots, n\}$ be the set of free nodes with $|\mathcal{V}_f| = n_f = n - n_a$, whose positions, denoted by $p_f = [p_{n_a+1}^\top, \ldots, p_n^\top]^\top \in \mathbb{R}^{2n_f}$, are to be determined. We assume that no overlapping points exist in $p = [p_a^\top, p_f^\top]^\top \in \mathbb{R}^{2n}$. Let I_2, 1_n, $\otimes$, $\lambda_{\max}$, and $\lambda_{\min}$ be the 2-by-2 identity matrix, $n \times 1$ column vector of all ones, the Kronecker product, the maximum eigenvalue, and the minimum eigenvalue of a symmetric real matrix, respectively. Denote by $\bar{R}(\theta)$ the 2D rotation matrix with rotation angle θ.

2.2. Angle measurements

Let $\sum_g$ be the fixed global coordinate frame, and $\sum_i$ be the node i's local coordinate frame for angle measurements, where $i \in \mathcal{V}_a \cup \mathcal{V}_f$. Let $Q_i \in SO(2)$ be the unknown rotation matrix from $\sum_g$ to $\sum_i$. Define the bearing from node i to node j, $j \in \mathcal{V}_a \cup \mathcal{V}_f$ in $\sum_g$ by $b_{ij} := (p_j - p_i)/\|p_j - p_i\|$, and in $\sum_i$ by $b_{ij}^i := (p_j^i - p_i^i)/\|p_j^i - p_i^i\| = Q_i b_{ij}$, where p_j^i represents the node j's coordinate in $\sum_i$. Assume that each node i measures the angle $\alpha_{kij} \in [0, 2\pi)$ with respect to

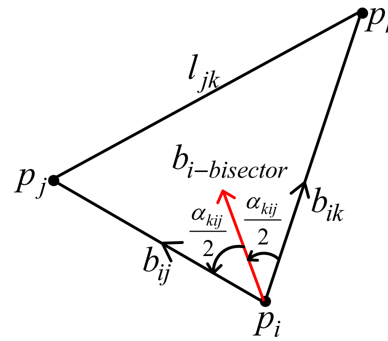


Fig. 1. Three sensor nodes k, i, j form a triangle.

its neighboring nodes $k, j \in \mathcal{V}_a \cup \mathcal{V}_f$ under the counterclockwise direction, which can be calculated by (Chen et al., 2021)

$$\alpha_{kij} := \begin{cases} \arccos(b_{ij}^\top b_{ik}), & \text{if } b_{ij}^\top b_{ik}^\perp \geq 0, \\ 2\pi - \arccos(b_{ij}^\top b_{ik}), & \text{otherwise}, \end{cases} \tag{1}$$

where $b_{ik}^\perp := \bar{R}(\frac{\pi}{2}) b_{ik} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} b_{ik}$, $j, k \in \mathcal{N}_i$, and $\mathcal{N}_i$ is node i's neighbor set. Since $b_{ij}^{i\top} b_{ik}^{i\perp} = b_{ij}^\top b_{ik}^\perp$ and $b_{ij}^{i\top} b_{ik}^i = b_{ij}^\top b_{ik}$, the angle measurement α_{kij} is independent of $\sum_i$.

2.3. Triangular angularity and triangular trigraph

First, we recall the definition of angularity from Chen et al. (2021), which is more efficient than using a graph to describe a network with triple-vertex angle constraints. For the vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$, define a three-vertex *triplet* (k, i, j) to describe the angle constraint α_{kij}. Then, we define $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V} \times \mathcal{V} = \{(k, i, j), k, i, j \in \mathcal{V}, i \neq j \neq k\}$ as an angle set, each element of which is a triplet. Since constraining α_{kij} is equivalent to constraining α_{jik}, we allow (j, i, k) to freely change to (k, i, j) for a given $\mathcal{A}$. Then, the combination of the vertex set $\mathcal{V}$, the angle set $\mathcal{A}$ and the position configuration $p \in \mathbb{R}^{2n}$ is called an *angularity* which we denote by $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$. Without p, the combination of the vertex set $\mathcal{V}$ and the angle set $\mathcal{A}$ is called a *trigraph* $\mathcal{T}(\mathcal{V}, \mathcal{A})$. We say $\mathcal{A}$ is a *triangular angle set* if for every $(i_1, j_1, k_1) \in \mathcal{A}$, there also exists $\{(j_1, k_1, i_1), (k_1, i_1, j_1)\} \subset \mathcal{A}$. Then, a triangular angle set $\mathcal{A}$ can be written in the form of

$$\mathcal{A} = \{\ldots, (i_1, j_1, k_1), (j_1, k_1, i_1), (k_1, i_1, j_1), \ldots\}. \tag{2}$$

We say $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is a *triangular angularity* and $\mathcal{T}(\mathcal{V}, \mathcal{A})$ a *triangular trigraph* if $\mathcal{A}$ is a triangular angle set. Denote by $m(\mathcal{T}) \in \mathbb{N}^+$ the total number of triangles in $\mathcal{T}$.

3. Triangular angle rigidity

In this section, we first establish an angle-induced linear constraint, and then develop triangular angle rigidity, and finally discuss a triangular angle rigidity matrix.

3.1. Angle-induced linear constraint in a triangle

Consider three non-collinear nodes k, i, j forming a triangle in Fig. 1. Denote by $b_{i-bisector} \in \mathbb{R}^{2\times 1}$ the bearing vector starting from p_i and pointing towards the bisector of the interior angle α_{kij}. Different from Lin et al. (2016) where local bearing measurements within a triangle are used to establish a complex linear constraint, we establish the angle-induced linear constraint using the fact

$$b_{i-bisector} = \bar{R}(-\frac{\alpha_{kij}}{2})\frac{p_j - p_i}{l_{ij}} = \bar{R}(\frac{\alpha_{kij}}{2})\frac{p_k - p_i}{l_{ik}}, \tag{3}$$

where $l_{ij} := \|p_i - p_j\|$ and $\bar{R}(-\alpha_{kij}/2) = \bar{R}^\top(\alpha_{kij}/2)$. Because the distance information l_{ij}, l_{ik} in (3) cannot be obtained from angle measurements, we use the law of sines in $\triangle ijk$ to describe their relative magnitude.[1] Multiplying $\bar{R}^\top(\alpha_{kij}/2)l_{ij}$ in both sides of (3) yields

$$\begin{aligned}&\bar{R}^\top(\frac{\alpha_{kij}}{2})l_{ij}\left(\bar{R}^\top(\frac{\alpha_{kij}}{2})\frac{p_j-p_i}{l_{ij}} - \bar{R}(\frac{\alpha_{kij}}{2})\frac{p_k-p_i}{l_{ik}}\right)\\ =&\bar{R}^\top(\alpha_{kij})(p_j-p_i) - \frac{\sin\alpha_{jki}}{\sin\alpha_{ijk}}(p_k-p_i) = 0.\end{aligned} \tag{4}$$

We say (4) is an *angle-induced linear constraint* in triangle $\triangle ijk$. Since the coefficient matrices in front of p_i, p_j, p_k in (4) are only related to α_{kij}, α_{jki} and α_{ijk}, (4) can be established among nodes i, j and k using their angle measurements and inter-node communication. Although Fig. 1 uses the case $\alpha_{kij} \in (0, \pi)$, it can be verified that (4) also holds for the case $\alpha_{kij} \in (\pi, 2\pi)$. By similarly defining $b_{j-bisector}$ and using (3), one also has another angle-induced linear constraint in $\triangle ijk$

$$\bar{R}^\top(\alpha_{ijk})(p_k - p_j) - (\sin\alpha_{kij}/\sin\alpha_{jki})(p_i - p_j) = 0. \tag{5}$$

Now, we present the relationship between (4) and (5).

Lemma 1. *The angle-induced linear constraints (4) and (5) in $\triangle ijk$ are interchangeable/linearly dependent.*

The proof of Lemma 1 is given in Chen (2022). Lemma 1 implies that each triangle only has one independent angle-induced linear constraint. In contrast to (4), we now introduce how to obtain the values of the three interior angles from a linear constraint among three nodes.

Lemma 2. *If three unknown points p_i, p_j, p_k satisfy $\bar{R}(\theta)(p_j - p_i) - \varepsilon_{ik}(p_k - p_i) = 0$ where $\bar{R}(\theta) \in SO(2)$ and ε_{ik} is a nonzero scalar, then the three interior angles in $\triangle ijk$ are uniquely determined.*

The proof of Lemma 2 is also given in Chen (2022). Compared with Fang et al. (2020, Thm 1), the constraints (4) and (5) are linear.

3.2. Triangular angle rigidity

For the triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, denote by $\alpha^* = [\ldots, \alpha^*_{ijk}, \alpha^*_{jki}, \alpha^*_{kij}, \ldots]^\top \in \mathbb{R}^{|\mathcal{A}|}$, $(i, j, k) \in \mathcal{A}$ those constant angle constraints defined by $\mathcal{A}$, and p position variables. Given $(i, j, k) \in \mathcal{A}$, define a matrix-weighted vector function of p_i, p_j, p_k as

$$f_i^{\triangle ijk}(\alpha^*, p) := A_i^{\triangle ijk}(\alpha^*)p_i + A_j^{\triangle ijk}(\alpha^*)p_j + A_k^{\triangle ijk}(\alpha^*)p_k, \tag{6}$$

where $f_i^{\triangle ijk} \in \mathbb{R}^{2\times 1}$, and the coefficient matrices

$$A_i^{\triangle ijk}(\alpha^*) := \left(\sin\alpha^*_{jki}I_2 - \sin\alpha^*_{ijk}\bar{R}^\top(\alpha^*_{kij})\right) \in \mathbb{R}^{2\times 2},$$
$$A_j^{\triangle ijk}(\alpha^*) := \sin\alpha^*_{ijk}\bar{R}^\top(\alpha^*_{kij}) \in \mathbb{R}^{2\times 2},$$
$$A_k^{\triangle ijk}(\alpha^*) := -\sin\alpha^*_{jki}I_2 \in \mathbb{R}^{2\times 2}$$

are defined according to (4), which satisfy $A_i^{\triangle ijk}(\alpha^*) + A_j^{\triangle ijk}(\alpha^*) + A_k^{\triangle ijk}(\alpha^*) \equiv 0$. If $\alpha^*_{ijk}, \alpha^*_{jki}, \alpha^*_{kij}$ are calculated under p, then $f_i^{\triangle ijk}(\alpha^*(p), p) = 0$. In contrast to (4), (6) is well-defined even when $\sin\alpha^*_{ijk} = 0$.

Remark 1. The reason of constructing such form of $f_i^{\triangle ijk}(\alpha^*, p)$ is that for the static sensor network, those angle measurements are known and constant, but the nodes' positions are unknown and to be determined.

We say $\triangle ijk$ is *strongly similar* to $\triangle i'j'k'$ if $\triangle ijk$ is similar to $\triangle i'j'k'$ and no reflection is between them, which we denote by $\triangle ijk \simeq \triangle i'j'k'$.

Lemma 3. *For non-collinear $p = [p_i^\top, p_j^\top, p_k^\top]^\top$ and $p' = [p_i'^\top, p_j'^\top, p_k'^\top]^\top$, if $f_i^{\triangle ijk}(\alpha^*(p), p') = A_i^{\triangle ijk}(\alpha^*(p))p_i' + A_j^{\triangle ijk}(\alpha^*(p))p_j' + A_k^{\triangle ijk}(\alpha^*(p))p_k' = 0$ where $A_i^{\triangle ijk}(\alpha^*(p))$ is a 2-by-2 matrix whose angles α^* are calculated under p, then $\triangle ijk \simeq \triangle i'j'k'$.*

Proof. Since the three points in p are non-collinear, using the definitions of $A_i^{\triangle ijk}, A_j^{\triangle ijk}, A_k^{\triangle ijk}$, and $f_i^{\triangle ijk}(\alpha^*(p), p')$, one has $\bar{R}^\top(\alpha^*_{kij}(p))(p_j' - p_i') - \frac{\sin\alpha^*_{jki}(p)}{\sin\alpha^*_{ijk}(p)}(p_k' - p_i') = 0$. Using Lemma 2, one has $\alpha^*_{jki}(p) = \alpha_{jki}(p')$, $\alpha^*_{jki}(p) = \alpha_{jki}(p')$ and $\alpha^*_{jki}(p) = \alpha_{jki}(p')$. Since each angle defined by (1) is calculated under a specific direction, $\triangle ijk$ is strongly similar to $\triangle i'j'k'$. □

Lemma 3 provides an algebraic approach to check triangles' similarities. Now, we define triangular equivalence and triangular congruence.

Definition 1. A triangular angularity $\mathbb{A}'(\mathcal{V}, \mathcal{A}, p')$ is triangularly equivalent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ if $f_i^{\triangle ijk}(\alpha^*(p), p') = 0$ for all $(i, j, k) \in \mathcal{A}$.

Definition 2. A triangular angularity $\mathbb{A}'(\mathcal{V}, \mathcal{A}, p')$ is triangularly congruent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ if $f_i^{\triangle ijk}(\alpha^*(p), p') = 0$ for all $i, j, k \in \mathcal{V}$, $i \neq j \neq k$.

When p is generic,[2] Definition 1 (resp. Definition 2) implies that $\mathbb{A}$ and $\mathbb{A}'$ are triangularly equivalent (resp. congruent) if their triangles defined in $\mathcal{A}$ (resp. all possible triangles) are correspondingly strongly similar.

Definition 3 (*Chen et al., 2021*)**.** Triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is triangularly angle rigid if there exists a constant $\varepsilon > 0$ such that any triangular angularity $\mathbb{A}'(\mathcal{V}, \mathcal{A}, p')$ which is triangularly equivalent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ and satisfies $\|p - p'\| < \varepsilon$ is also triangularly congruent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$.

Definition 4 (*Chen et al., 2021*)**.** Triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is globally triangularly angle rigid if any triangular angularity $\mathbb{A}'(\mathcal{V}, \mathcal{A}, p')$ which is triangularly equivalent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is also triangularly congruent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$.

Different from using angles separately to define angle rigidity in Chen et al. (2021, Def 3), triangular angle rigidity defined in this paper uses *one* matrix-weighted vector (6) to describe *three* interior angles associated within one triangle. Similar to Chen et al. (2021, Fig. 3), non-generic embedding of $\mathcal{T}$ also results in different properties than the triangular angularity with generic configuration p. For example, any triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with collinear configuration p and arbitrary nonempty $\mathcal{A}$ is globally triangularly angle rigid. However, the probability that a randomly chosen configuration p is generic is 1 (Connelly & Guest, 2015, Thm 7.2.1), i.e., the density of non-generic configurations is measure zero in the whole Euclidean space. Therefore, we are more interested in the properties of the triangular angularity with generic configuration. Because triangular angle rigidity of $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with generic p is a property of the trigraph $\mathcal{T}(\mathcal{V}, \mathcal{A})$, we also say that a trigraph $\mathcal{T}(\mathcal{V}, \mathcal{A})$ is *(generically) triangularly angle rigid* if $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is triangularly angle rigid for generic configurations p. Then, we have the relationship between angle rigidity and triangular angle rigidity.

[1] This becomes invalid when p_i, p_j, p_k are collinear.

[2] The definition of generic here is the same as Chen et al. (2021, Def 4).

Lemma 4. *A (resp. globally) triangularly angle rigid angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with generic configuration p is (resp. globally) angle rigid.*

Proof. Note that if p is generic, none of three points in p will be collinear. According to Lemma 3, if $f_i^{\triangle ijk}(\alpha^*(p), p') = 0$, then $\triangle ijk \simeq \triangle i'j'k'$ which implies $\alpha_{ijk}(p) = \alpha_{ijk}(p')$, $\alpha_{jki}(p) = \alpha_{jki}(p')$, and $\alpha_{kij}(p) = \alpha_{kij}(p')$. Therefore, $f_i^{\triangle ijk}(\alpha^*(p), p') = 0$ used in Definitions 1 and 2 is equivalent to constraining the three angles separately, which is the case in the definition of angle rigidity (Chen et al., 2021, Def 3). This equivalence implies that the triangularly angle rigid angularity $\mathbb{A}$ is angle rigid. The same case holds for global angle rigidity. $\square$

The difference between triangular angle rigidity and angle rigidity is that the angle set $\mathcal{A}$ in triangular angle rigidity must be a triangular angle set, while in angle rigidity it can be non-triangular.

3.3. Triangular angle rigidity matrix

Since p are variables in (6), for the triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, we can define the *triangular angle function*

$$f_{\mathcal{A}}(p) := [\ldots, (f_{i_1}^{\triangle i_1 j_1 k_1}(\alpha^*, p))^\top, \ldots]^\top \in \mathbb{R}^{2m(\mathcal{T})},$$

where $(i_1, j_1, k_1) \in \mathcal{A}$, and α^* in $f_{i_1}^{\triangle i_1 j_1 k_1}(\alpha^*, p)$ represents those constant angle constraints associated with $\triangle i_1 j_1 k_1$. Using the Taylor series expansion, one has

$$f_{\mathcal{A}}(p + \delta p) = f_{\mathcal{A}}(p) + R_{\mathcal{A}}(\alpha^*)\delta p + \text{high order terms}, \tag{7}$$

where $R_{\mathcal{A}}(\alpha^*) := \frac{\partial f_{\mathcal{A}}(p)}{\partial p} \in \mathbb{R}^{2m(\mathcal{T})\times 2n}$ is defined as the *triangular angle rigidity matrix*, and δp is the infinitesimal motion of p. Because all the coefficient matrices in (6) are constant matrices, $R_{\mathcal{A}}(\alpha^*)$ can be written by

$$\begin{array}{c} \\ \text{1st } \triangle \\ \cdots \\ \triangle ijk \\ \cdots \\ m(\mathcal{T})\text{th } \triangle \end{array}
\begin{array}{c} \begin{array}{ccccccc} \cdots & \text{Vertex } i & \cdots & \text{Vertex } j & \cdots & \text{Vertex } k & \cdots \end{array} \\ \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & A_i^{\triangle ijk} & 0 & A_j^{\triangle ijk} & 0 & A_k^{\triangle ijk} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array}$$

whose row blocks are indexed by the triangles in $\mathcal{A}$ and column blocks the vertices. Different from the angle rigidity matrix defined in Chen et al. (2021, Eqn 13), the triangular angle rigidity matrix $R_{\mathcal{A}}(\alpha^*)$ is only related to the values of those constrained angles α^* in $\mathcal{A}$ but not related to the sensor nodes' position information p or inter-node distance information, which plays an important role in this network localization problem. Define $\mathcal{A}^* := \{(i, j, k), \forall i, j, k \in \mathcal{V}, i \neq j \neq k\}$ as the complete angle set, and $R_{\mathcal{A}^*}(\bar{\alpha}^*)$ and $R_{\mathcal{A}}(\alpha^*)$ as the triangular angle rigidity matrices of triangular angularities $\mathbb{A}(\mathcal{V}, \mathcal{A}^*, p)$ and $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, respectively, where $\bar{\alpha}^* = [\ldots, \alpha_{ijk}^*, \alpha_{jki}^*, \alpha_{kij}^*, \ldots]^\top \in \mathbb{R}^{|\mathcal{A}^*|}$, $(i, j, k) \in \mathcal{A}^*$.

Theorem 1. *For the triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with generic p, one has* $\mathrm{Span}\{1_n \otimes I_2,\ \left(I_n \otimes \bar{R}(\frac{\pi}{2})\right) p,\ p\} \subseteq \mathrm{Null}(R_{\mathcal{A}^*}(\bar{\alpha}^*(p))) \subseteq \mathrm{Null}(R_{\mathcal{A}}(\alpha^*(p)))$*, and* $\mathrm{Rank}(R_{\mathcal{A}}(\alpha^*(p))) \leq \mathrm{Rank}(R_{\mathcal{A}^*}(\bar{\alpha}^*(p))) \leq 2n - 4$.

The proof of Theorem 1 is given in Appendix A. We now discuss the relationship between triangular angle rigidity and global triangular angle rigidity.

Theorem 2. *A triangular angularity $\mathbb{A}'(\mathcal{V}, \mathcal{A}, p')$ is triangularly equivalent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ if and only if $R_{\mathcal{A}}(\alpha^*(p))\, p' = 0$. Also, $\mathbb{A}'$ is triangularly congruent to $\mathbb{A}$ if and only if $R_{\mathcal{A}^*}(\bar{\alpha}^*(p))\, p' = 0$.*

Proof. Note that in $R_{\mathcal{A}}(\alpha^*(p))$, the angles α^* are calculated under p. According to Definitions 1 and 2 and the structure of the triangular angle rigidity matrix $R_{\mathcal{A}}$, the conclusion can be obtained straightforwardly by writing all the triangles' linear constraints into a compact form. $\square$

Lemma 5. *A triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is globally triangularly angle rigid if and only if* $\mathrm{Null}(R_{\mathcal{A}^*}(\bar{\alpha}^*(p))) = \mathrm{Null}(R_{\mathcal{A}}(\alpha^*(p)))$*, or equivalently* $\mathrm{Rank}(R_{\mathcal{A}^*}(\bar{\alpha}^*(p))) = \mathrm{Rank}(R_{\mathcal{A}}(\alpha^*(p)))$.

The proof of Lemma 5 is given in Chen (2022), which can also be obtained by following Zhao and Zelazo (2016a, Thm 2).

Theorem 3. *For a triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, it is globally triangularly angle rigid if and only if it is triangularly angle rigid.*

The proof of Theorem 3 is given in Appendix B. According to Chen et al. (2021, Thm 1), angle rigidity does not necessarily imply global angle rigidity. However, Theorem 3 implies that for triangular angle rigidity where angle constraints are associated within triangles in $\mathcal{A}$, triangular angle rigidity implies global triangular angle rigidity.

Theorem 4. *A trigraph $\mathcal{T}(\mathcal{V}, \mathcal{A})$ is triangularly angle rigid if and only if* $\mathrm{Rank}(R_{\mathcal{A}}(\alpha^*(p))) = 2n - 4$ *where p is an arbitrary generic configuration.*

The proof of Theorem 4 is given in Chen (2022), which can also be obtained by using Lemma 1 and Chen et al. (2021, Thm 2). Different from Chen et al. (2021) where the rank checking condition for angle rigidity is related to inter-node distances and bearings, the rank checking condition in Theorem 4 is only related to the interior angles. We develop localizability conditions for triangular angle-constrained sensor networks in the next section, where more results on triangular angle rigidity will be presented.

4. Localizability conditions

If $(i, j, k) \in \mathcal{A}$, then we say nodes i, j, k are neighboring nodes with one another, i.e., $\{j, k\} \subseteq \mathcal{N}_i$, $\{i, k\} \subseteq \mathcal{N}_j$, and $\{j, i\} \subseteq \mathcal{N}_k$. Now, we formulate the angle-only network localization problem.

Problem 1. Consider a 2D triangular sensor network described by $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with $\mathcal{V} = \mathcal{V}_a \cup \mathcal{V}_f$, $n_a \geq 2$, and generic p. Given the anchor nodes' positions p_a in $\sum_g$, the aim is to determine the free nodes' positions p_f using the nodes' angle measurements and inter-node communication, whose topologies are described by triangular trigraph $\mathcal{T}(\mathcal{V}, \mathcal{A})$.

Denote by $\hat{p} = [\hat{p}_a^\top, \hat{p}_f^\top]^\top$ the estimation of all nodes' positions. Since one triangle's three angle constraints in $\mathcal{A}$ will give one angle-induced linear constraint, the localization Problem 1 is equivalent to finding $\hat{p}_f$ subject to

$$\begin{aligned} f_i^{\triangle ijk}(\alpha^*(p), \hat{p}) &= A_i^{\triangle ijk}(\alpha^*(p))\hat{p}_i + A_j^{\triangle ijk}(\alpha^*(p))\hat{p}_j \\ &+ A_k^{\triangle ijk}(\alpha^*(p))\hat{p}_k = 0, \quad \hat{p}_i = p_i, \quad \forall i \in \mathcal{V}_a, \end{aligned} \tag{8}$$

where $(i, j, k) \in \mathcal{A}$, and $A_i^{\triangle ijk}, A_j^{\triangle ijk}, A_k^{\triangle ijk}$ are constant matrices since the angles α^* and p are constants.

Definition 5. The triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is said to be *localizable* if the solution $\hat{p}_f$ to the problem (8) is globally unique and $\hat{p}_f = p_f$.

Next, we introduce both algebraic and topological localizability conditions for the triangular sensor network $\mathbb{A}$.

4.1. Algebraic localizability condition

Different from Jing et al. (2022) where the localization problem is formulated as a nonlinear optimization problem, we transfer the localization problem (8) into a linear least-square optimization problem by defining the cost function

$$J(\hat{p}) = \sum_{(i,j,k)\in\mathcal{A}} \|A_i^{\triangle ijk}\hat{p}_i + A_j^{\triangle ijk}\hat{p}_j + A_k^{\triangle ijk}\hat{p}_k\|^2, \tag{9}$$

where $\hat{p}_i = p_i, \forall i \in \mathcal{V}_a$. Then, we want to know under which condition the true position p_f is the unique and global minimizer of (9), which is the localizability condition. According to the definition of the triangular angle rigidity matrix, one has

$$J(\hat{p}) = \hat{p}^\top R_{\mathcal{A}}^\top(\alpha^*(p))R_{\mathcal{A}}(\alpha^*(p))\hat{p}. \tag{10}$$

Let $D(\alpha^*) := R_{\mathcal{A}}^\top(\alpha^*)R_{\mathcal{A}}(\alpha^*) \in \mathbb{R}^{2n\times 2n}$. By partitioning matrix $R_{\mathcal{A}} = [R_{\mathcal{A}}^a \;\; R_{\mathcal{A}}^f]$ into anchor nodes' part $R_{\mathcal{A}}^a \in \mathbb{R}^{2m(\mathcal{T})\times 2n_a}$ and free nodes' part $R_{\mathcal{A}}^f \in \mathbb{R}^{2m(\mathcal{T})\times 2n_f}$, the matrix $D(\alpha^*)$ can be written in the form of $D(\alpha^*) = \begin{bmatrix} D_{aa} & D_{af} \\ D_{fa} & D_{ff}\end{bmatrix}$, where $D_{aa} = (R_{\mathcal{A}}^a)^\top R_{\mathcal{A}}^a \in \mathbb{R}^{2n_a\times 2n_a}$, $D_{af} = (R_{\mathcal{A}}^a)^\top R_{\mathcal{A}}^f \in \mathbb{R}^{2n_a\times 2n_f}$, $D_{fa} = (R_{\mathcal{A}}^f)^\top R_{\mathcal{A}}^a \in \mathbb{R}^{2n_f\times 2n_a}$, and $D_{ff} = (R_{\mathcal{A}}^f)^\top R_{\mathcal{A}}^f \in \mathbb{R}^{2n_f\times 2n_f}$.

Lemma 6. *If $\hat{p}_f^*$ is a minimizer of the cost function (9), then it is also a global minimizer and $D_{ff}\hat{p}_f^* + D_{fa}p_a = 0$.*

Proof. Substituting the matrix $D(\alpha^*)$ into (10) yields

$$J(\hat{p}) = \tilde{J}(\hat{p}_f) = p_a^\top D_{aa}p_a + 2p_a^\top D_{af}\hat{p}_f + \hat{p}_f^\top D_{ff}\hat{p}_f, \tag{11}$$

where we used $\hat{p}_a = p_a$. It follows that any minimizer of (11) satisfies $\nabla_{\hat{p}_f^*}\tilde{J}(\hat{p}_f^*) = D_{ff}\hat{p}_f^* + D_{fa}p_a = 0$. Then, by following the same line as Zhao and Zelazo (2016b, Lem 4), $\hat{p}_f^*$ is a global minimizer. □

Theorem 5. *A triangular angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ with $n_a \geq 2$ and generic p is localizable if and only if D_{ff} is nonsingular. When the angularity is localizable, the true positions of the free nodes can be calculated by $p_f = -D_{ff}^{-1}D_{fa}p_a$.*

The proof of Theorem 5 can be straightforwardly obtained by using Lemma 6. The algebraic localizability condition in Theorem 5 is more straightforward than those in Jing et al. (2022). However, the algebraic localizability condition depends on all the angle measurements. Next, we develop topological localizability condition which does not depend on the sensor nodes' angle measurements but only depends on the topology $\mathcal{T}(\mathcal{V},\mathcal{A})$.

4.2. Topological localizability condition

Based on Theorem 4, we first show the relationship between the network localizability and triangular angle rigidity.

Theorem 6. *For a triangular angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ with $n_a = 2$ and generic p, it is localizable if and only if the trigraph $\mathcal{T}(\mathcal{V},\mathcal{A})$ is triangularly angle rigid.*

The proof of Theorem 6 is given in Appendix C. Theorem 6 requires that the number of anchor nodes is 2. We give an example in Fig. 2 to show that when $n_a = 3$, the necessity of Theorem 6 does not hold. However, the sufficient part of Theorem 6 still holds for $n_a \geq 2$. The network in Fig. 2 is unlocalizable under (Jing et al., 2022) since the angle in Jing et al. (2022) is defined without a direction (its magnitude is in $[0,\pi]$), which also indicates that the localizability condition in Theorem 6 is milder than those in Jing et al. (2022, Thm 2).

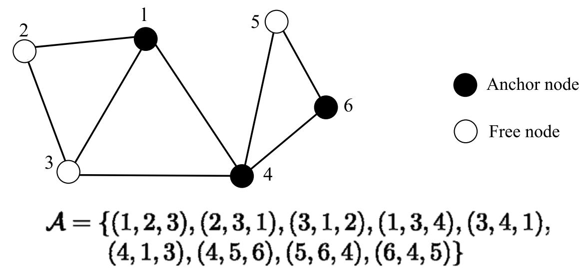


Fig. 2. A localizable but triangularly non-angle rigid angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ with $n_a = 3$.

Although Theorem 6 proposes a rigidity-based condition to check the localizability, the available checking condition of trigraph $\mathcal{T}$'s generic triangular angle rigidity still relies on algebraic information according to Theorem 4. Similar cases exist in those localizability conditions using bearing rigidity (Zhao & Zelazo, 2016b, Thm 3 and Lem 2) and angle-displacement rigidity (Fang et al., 2020, Thm 6 and Thm 2). Some pure topological localizability conditions have been proposed in Jing et al. (2022, Corollary 2) and Lin et al. (2016, Thm 4.1), which are only sufficient. Different from these previous works, we aim to propose a pure topological, necessary and sufficient localizability condition by developing a topological, necessary and sufficient checking condition for $\mathcal{T}$'s generic triangular angle rigidity. This condition is inspired by Laman's theorem (Laman, 1970, Thm 6.5) which is a classic result on generic distance rigidity and has played a very important role in the development of rigidity graph theory during the past fifty years (Connelly & Guest, 2015; Whiteley, 1996). Before giving the condition, we first present some related definitions.

Definition 6. A triangular trigraph $\mathcal{T}(\mathcal{V},\mathcal{A})$ is minimally and triangularly angle rigid if $\mathcal{T}$ is triangularly angle rigid and the number of triangles $m(\mathcal{T}) = n-2$.

From Definition 6, the minimum number of triangles in $\mathcal{A}$ to make a triangular trigraph $\mathcal{T}$ angle rigid is $n-2$, which can be seen from Theorem 4 and the definition of $R_{\mathcal{A}}$. Inspired by Laman's theorem and its proof Laman (1970, Thm 5.6), we define a special type of trigraph.

Definition 7. A trigraph $\mathcal{T}(\mathcal{V},\mathcal{A})$ is said to be a L-trigraph if it satisfies the property L: (a) $\mathcal{T}$ is a triangular trigraph, (b) the number of triangles $m(\mathcal{T}) = |\mathcal{V}|-2$, and (c) for any subset $\mathcal{V}'$ of $\mathcal{V}$, the induced triangular subtrigraph $\mathcal{T}'(\mathcal{V}',\mathcal{A}')$ of $\mathcal{T}$ satisfies $m(\mathcal{T}') \leq |\mathcal{V}'|-2$.

Now, we present a fact which will be an important foundation for the follow-up analysis. Note that in an infinitesimally and minimally distance rigid graph, there must exist a vertex associated with fewer than 4 edges (Laman, 1970, Prop 6.1). We also have a similar conclusion for minimally and triangularly angle rigid trigraphs.

Lemma 7. *For a minimally and triangularly angle rigid trigraph $\mathcal{T}(\mathcal{V},\mathcal{A})$, there must exist a vertex associated with 1 or 2 triangles in $\mathcal{A}$.*

The proof of Lemma 7 is given in Appendix D. Now, we present the topological, necessary and sufficient condition for generic triangular angle rigidity.

Theorem 7. *A triangular trigraph $\mathcal{T}(\mathcal{V},\mathcal{A})$ is triangularly angle rigid in 2D if and only if there exists a subtrigraph $\mathcal{T}'(\mathcal{V},\mathcal{A}')$ with $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{T}'$ being a L-trigraph.*

The proof of Theorem 7 is given in Appendix E. Theorem 7 implies that a trigraph $\mathcal{T}$ is minimally and triangularly angle rigid if and only if it is a L-trigraph. Combining Theorems 6 and 7 yields the topological, necessary and sufficient localizability condition for a triangular network $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with $n_a = 2$ and generic p.

Remark 2. In Appendix E, three types of triangle deletion operations are defined. One can also define the reverse of the triangle deletion operations as triangle addition operations, in which $\mathcal{T}_0$ adds one triangle constraint, and then becomes $\mathcal{T}'$. Note that these three types of triangle addition operations in triangular angle rigidity plays a similar role as the Henneberg construction in distance rigidity (Henneberg, 1911). In addition, different from the case in distance rigidity, the condition developed in Theorem 7 is also a topological, necessary and sufficient condition for generic global triangular angle rigidity according to Theorem 3.

5. Distributed localization

In this section, we design both continuous and discrete localization algorithms to achieve $\hat{p}_f \to p_f$.

5.1. Continuous localization algorithm

Based on the least-square optimization problem (9), we design a gradient descent localization algorithm

$$\dot{\hat{p}}_f(t) = -\nabla_{\hat{p}_f}\tilde{J}(\hat{p}_f) = -D_{ff}\hat{p}_f(t) - D_{fa}p_a, \tag{12}$$

whose component form for each free node is

$$\begin{aligned}\dot{\hat{p}}_i(t) = &-\sum_{(i,j_1,k_1)\in\bar{\mathcal{A}}} (A_i^{\triangle ij_1k_1})^\top f_i^{\triangle ij_1k_1}(\alpha^*, \hat{p}(t))\\ &-\sum_{(j_2,i,k_2)\in\bar{\mathcal{A}}} (A_i^{\triangle j_2ik_2})^\top f_i^{\triangle j_2ik_2}(\alpha^*, \hat{p}(t))\\ &-\sum_{(j_3,k_3,i)\in\bar{\mathcal{A}}} A_i^{\triangle j_3k_3i} f_i^{\triangle j_3k_3i}(\alpha^*, \hat{p}(t)),\ \ i\in\mathcal{V}_f,\end{aligned} \tag{13}$$

where $\hat{p}_j(t) = p_j, \forall j \in \mathcal{V}_a$, $f_i^{\triangle j_2ik_2}(\alpha^*, \hat{p}) = A_{j_2}^{\triangle j_2ik_2}\hat{p}_{j_2} + A_i^{\triangle j_2ik_2}\hat{p}_i + A_{k_2}^{\triangle j_2ik_2}\hat{p}_{k_2}$, $f_i^{\triangle j_3k_3i}(\alpha^*, \hat{p}) = A_{j_3}^{\triangle j_3k_3i}\hat{p}_{j_3} + A_{k_3}^{\triangle j_3k_3i}\hat{p}_{k_3} + A_i^{\triangle j_3k_3i}\hat{p}_i$, and $\bar{\mathcal{A}} \subset \mathcal{A}$, $|\bar{\mathcal{A}}| = m(\mathcal{T})$, and if $(i, j, k) \in \bar{\mathcal{A}}$, then $\{(j, i, k), (i, k, j)\} \not\subseteq \bar{\mathcal{A}}$. The distributed law (13) can be implemented by using node i's angle measurements to obtain $\alpha_{j_2ik_2}$, and inter-node communication to obtain $\alpha_{ik_1j_1}$, $\alpha_{j_3k_3i}$, $\hat{p}_{j_s}(t)$, $\hat{p}_{k_s}(t)$, $s = 1, 2, 3$. A detailed and specific form of (13) under a localizable network with eight sensor nodes is provided in Chen (2022).

Theorem 8. *If $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is triangularly angle rigid and p is generic, then Problem 1 is solved and $\hat{p}_f(t)$ globally and exponentially converges to p_f under the distributed and continuous localization algorithm (13).*

Proof. According to Section 4.2, the sensor network is localizable and D_{ff} is nonsingular and positive definite. Then, consider the candidate Lyapunov function $V_1(t) = 0.5\|p_f - \hat{p}_f(t)\|^2$ whose time-derivative is $\dot{V}_1(t) = -(p_f - \hat{p}_f(t))^\top \dot{\hat{p}}_f(t) \le -\lambda_{\min}(D_{ff})\|p_f - \hat{p}_f(t)\|^2$, where $p_f = -D_{ff}^{-1}D_{fa}p_a$. Since $\dot{V}_1(t)$ is negative definite, $\hat{p}_f(t)$ converges to p_f globally and exponentially. $\square$

To tune the convergence rate of the estimation error $\|\hat{p}_f - p_f\|$, a positive gain can be added in (12), i.e.,

$$\dot{\hat{p}}_f(t) = -k_c\nabla_{\hat{p}_f}\tilde{J}(\hat{p}_f) = -k_c(D_{ff}\hat{p}_f(t) + D_{fa}p_a), \tag{14}$$

where $k_c > 0$. Then, one has $\|p_i - \hat{p}_i(t)\| \le \|p_f - \hat{p}_f(t)\| = \sqrt{2V_1(t)} \le \sqrt{2V_1(0)}e^{-k_c\lambda_{\min}(D_{ff})t}$. In practice, angle measurements are subjected to noises, whose effects on the estimation error $\|\hat{p}_f - p_f\|$ can be similarly obtained by following Lin et al. (2016, Thm 4.3).

5.2. Discrete localization algorithm

Consider that the network localization law (12) is executed under discrete iteration dynamics. We define the constant sampling period as $h > 0$ and use the forward Euler approximation to describe the differential operation in the continuous algorithm (12). More specifically,

$$\dot{\hat{p}}_f(t)|_{t=kh} \approx (\hat{p}_f[k+1] - \hat{p}_f[k])/h, \quad k \in \mathbb{N}, \tag{15}$$

where $\hat{p}_f[k+1] = \hat{p}_f((k+1)h)$ and $\hat{p}_f[k] = \hat{p}_f(kh)$. Under (15), the continuous localization law (12) becomes

$$\hat{p}_f[k+1] = \hat{p}_f[k] - hD_{ff}\hat{p}_f[k] - hD_{fa}p_a, \tag{16}$$

where $\hat{p}_f[k] = \left[\hat{p}_{n_a+1}^\top[k], \dots, \hat{p}_n^\top[k]\right]^\top$. The component form of (16) can be described by

$$\begin{aligned}\hat{p}_i[k+1] = \hat{p}_i[k] &- h\sum_{(i,j_1,k_1)\in\bar{\mathcal{A}}} (A_i^{\triangle ij_1k_1})^\top f_i^{\triangle ij_1k_1}(\alpha^*, \hat{p}[k])\\ &- h\sum_{(j_2,i,k_2)\in\bar{\mathcal{A}}} (A_i^{\triangle j_2ik_2})^\top f_i^{\triangle j_2ik_2}(\alpha^*, \hat{p}[k])\\ &- h\sum_{(j_3,k_3,i)\in\bar{\mathcal{A}}} A_i^{\triangle j_3k_3i} f_i^{\triangle j_3k_3i}(\alpha^*, \hat{p}[k]).\end{aligned} \tag{17}$$

Defining $\tilde{p}_f[k] := \hat{p}_f[k] - p_f$, one has

$$\tilde{p}_f[k+1] = (I_{2n_f} - hD_{ff})\tilde{p}_f[k]. \tag{18}$$

Since D_{ff} is positive definite, all the eigenvalues of $(I_{2n_f} - hD_{ff})$ will be in the open unit disk if

$$h < 2\min_{i=1,\dots,2n_f} \lambda_i^{-1}(D_{ff}) = 2\lambda_{\max}^{-1}(D_{ff}). \tag{19}$$

Theorem 9. *If $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is triangularly angle rigid, p is generic, and the sampling period h satisfies (19), then Problem 1 is solved and $\hat{p}_f[k]$ globally converges to p_f under the discrete localization algorithm (17).*

The condition (19) can be satisfied in practice by, e.g., employing a distributed algorithm to estimate D_{ff}'s maximum eigenvalue (Lin et al., 2016) or properly using the information of each node's associated edges or triangles in the sensor network (Li et al., 2019).

Remark 3. Compared with the bearing-based localization (Bishop, Anderson, Fidan, Pathirana, & Mao, 2009; Eren et al., 2006; Li et al., 2019; Shames et al., 2012; Zhao & Zelazo, 2016b), the angle-based localization laws (13), (17) do not require the alignment of the nodes' coordinate frames. The required communication of the localization laws in Cao et al. (2021), Jing et al. (2022) and Lin et al. (2016) consists of measured local bearing vectors and estimated positions, while in (13), (17) only measured angles and estimated positions. Compared with the distributed localization in Jing et al. (2022) where the anchors must be neighboring and the localization topology is sequential, (13), (17) allow the anchors to be non-neighboring and the localization topology to be non-sequential.

6. Simulation examples

We use a sensor network with 2 anchors (labeled by 1 and 2) and 30 free nodes (labeled by 3 ∼ 32) to validate Theorem 7 and the localization laws (14) and (17). The network topology is given in Fig. 3, which consists of 30 triangles whose detailed forms are given in Chen (2022). It is verified that the trigraph in Fig. 3 is a L-trigraph. The fact $\text{Rank}(R_{\mathcal{A}}(\alpha^*(p))) = 60$ validates Theorem 7. According to Theorem 6, the network in Fig. 3 is localizable.

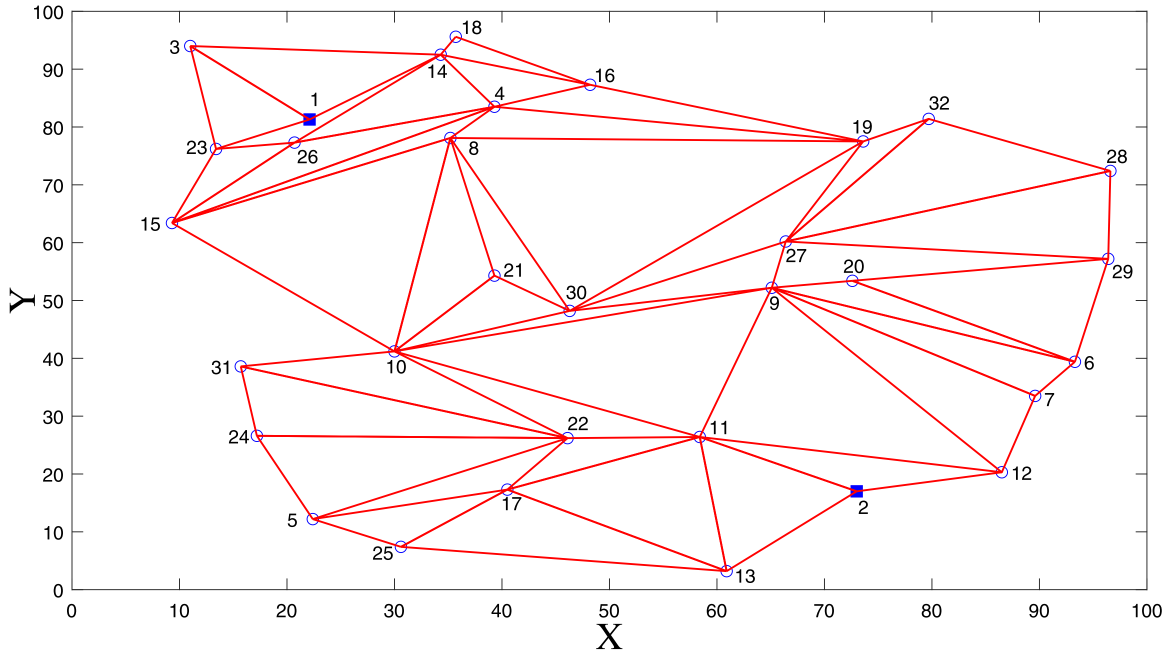


Fig. 3. Network topology with 32 nodes and 30 triangles.

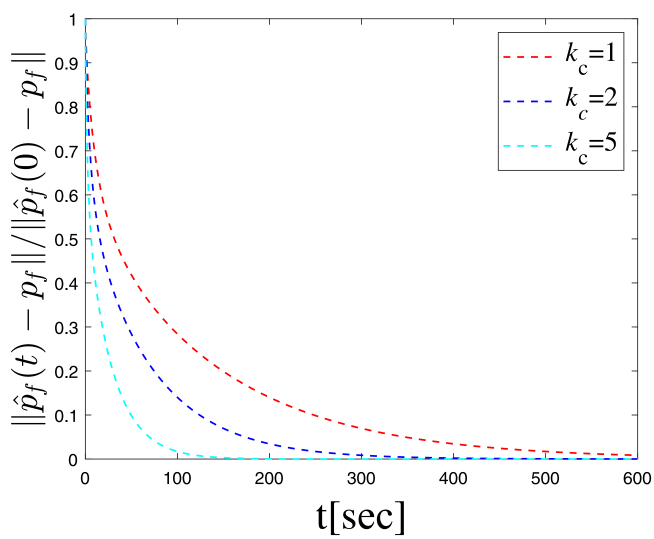


Fig. 4. Position estimation errors in continuous cases.

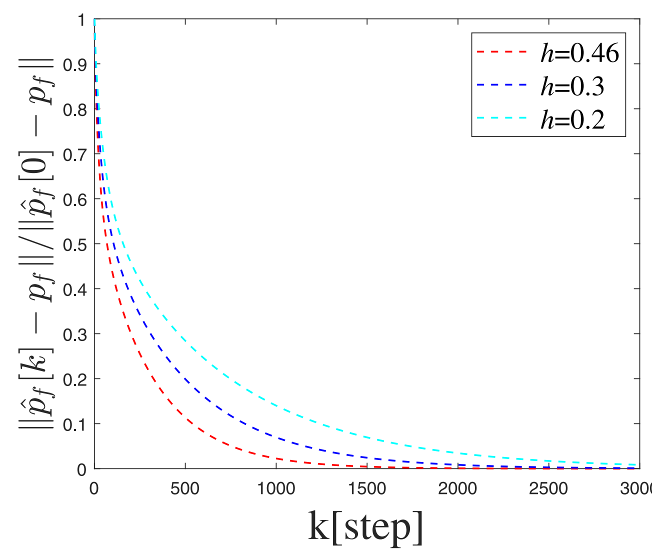


Fig. 5. Position estimation errors in discrete cases.

The changes of position estimation error $\|\hat{p}_f - p_f\|$ under the localization algorithms (14), (17) are shown in Figs. 4, 5, respectively. Fig. 4 shows that the convergence time is shorter when k_c is larger. Since $\lambda_{\max}(D_{ff}) \approx 4.33$, we need to choose $h < 0.462$. Fig. 5 shows that more iteration steps are needed for the convergence when h is smaller.

7. Conclusion

This paper has developed triangular angle rigidity for distributed localization using angle measurements in 2D. First, we have shown that triangular angle rigidity implies global triangular angle rigidity. We have proposed a topological, necessary and sufficient condition to check generic triangular angle rigidity, from which a trigraph is minimally and triangularly angle rigid if and only if it is a L-trigraph. Then, we have developed algebraic and topological localizability conditions, both of which can be necessary and sufficient. Moreover, both continuous and discrete distributed localization algorithms have been proposed, which only rely on the measured angles and estimated positions. Future work will focus on the 3D case, which cannot be obtained straightforwardly from this 2D case since when a 3D rotation matrix is used to establish a 3D angle-induced linear constraint like (4), the 3D rotation matrix will depend on its associated nodes' relative positions.

Appendix A. Proof of Theorem 1

The cases $\delta p = 1_n \otimes I_2$, $\delta p = \left(I_n \otimes \bar{R}(\pi/2)\right) p$ and $\delta p = p$ correspond to translation, rotation and scaling motion of $\mathbb{A}$, respectively. According to (6), one has $A_i^{\triangle ijk} + A_j^{\triangle ijk} + A_k^{\triangle ijk} \equiv 0$ which implies that $R_{\mathcal{A}}(\alpha^*(p))(1_n \otimes I_2) = 0$. According to (4), one has $f_i^{\triangle ijk}(\alpha^*(p), p) = 0$, which implies $R_{\mathcal{A}}(\alpha^*(p))p = 0$. Since

$A_i^{\triangle ijk}\bar{R}(\frac{\pi}{2}) = \bar{R}(\frac{\pi}{2})A_i^{\triangle ijk}$, one has $R_{\mathcal{A}}(\alpha^*(p))\left(I_n \otimes \bar{R}(\pi/2)\right)p = 0$. These three facts imply that $\{1_n \otimes I_2, \left(I_n \otimes \bar{R}(\pi/2)\right)p, p\}$ always lie in the null space of $R_{\mathcal{A}}$ and $R_{\mathcal{A}^*}$. Since $R_{\mathcal{A}}$ is a sub-matrix of $R_{\mathcal{A}^*}$ and they have the same number of columns, one has $\mathrm{Rank}(R_{\mathcal{A}}) \le \mathrm{Rank}(R_{\mathcal{A}^*})$ and $\mathrm{Null}(R_{\mathcal{A}^*}) \subseteq \mathrm{Null}(R_{\mathcal{A}})$. The independence of the four vectors in $\{1_n \otimes I_2, \left(I_n \otimes \bar{R}(\pi/2)\right)p, p\}$ can be similarly obtained by using Chen et al. (2021, Lem 2), which implies $\mathrm{Rank}(R_{\mathcal{A}^*}) \le 2n-4$. □

Appendix B. Proof of Theorem 3

The necessity of this theorem is straightforward by following Definitions 3 and 4. We now prove its sufficiency. Assume that $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ is triangularly angle rigid. According to Definition 3, any triangular angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ satisfying $R_{\mathcal{A}}(\alpha^*(p))p' = 0$ and $\|p-p'\| \le \varepsilon$ has that $R_{\mathcal{A}^*}(\bar{\alpha}^*(p))p' = 0$. Because $R_{\mathcal{A}}(\alpha^*(p))p = 0$ and $R_{\mathcal{A}^*}(\bar{\alpha}^*(p))p = 0$ using Theorem 1, one has that

$$R_{\mathcal{A}}(\alpha^*(p))\delta p = 0 \Rightarrow R_{\mathcal{A}^*}(\bar{\alpha}^*(p))\delta p = 0, \tag{20}$$

where $\delta p = p' - p$ and $\|\delta p\| \le \varepsilon$. Since $\delta p \in \mathbb{R}^{2n}$ is a vector, the constraint $\|\delta p\| \le \varepsilon$ allow δp to lie in a $2n$-dimensional ball with $\delta p = 0$ as its origin and ε as its radius. Therefore, $k_s\delta p$ will expand the entire Euclidean space when $k_s \in [0,\infty)$. Moreover, (20) implies that $R_{\mathcal{A}}(\alpha^*(p))(k_s\delta p) = k_s R_{\mathcal{A}}(\alpha^*(p))\delta p = 0 \Rightarrow R_{\mathcal{A}^*}(\bar{\alpha}^*(p))(k_s\delta p) = 0$ holds for an arbitrary $k_s \in [0,\infty)$. It follows that $\mathrm{Null}(R_{\mathcal{A}}(\alpha^*)) \subseteq \mathrm{Null}(R_{\mathcal{A}^*}(\bar{\alpha}^*))$. Since $\mathrm{Null}(R_{\mathcal{A}^*}(\bar{\alpha}^*)) \subseteq \mathrm{Null}(R_{\mathcal{A}}(\alpha^*))$ according to Theorem 1, one has that $\mathrm{Null}(R_{\mathcal{A}^*}(\bar{\alpha}^*)) = \mathrm{Null}(R_{\mathcal{A}}(\alpha^*))$, i.e., $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ is globally triangularly angle rigid. □

Appendix C. Proof of Theorem 6

Firstly, according to Theorem 5, $\mathbb{A}$ is localizable if and only if D_{ff} is nonsingular. Because $\mathrm{Rank}(D_{ff}) = \mathrm{Rank}(R^f_{\mathcal{A}})$ and $D_{ff} \in \mathbb{R}^{(2n-4)\times(2n-4)}$, one has that D_{ff} is nonsingular if and only if $\mathrm{Rank}(R^f_{\mathcal{A}}) = 2n-4$. Secondly, by Theorem 4, $\mathcal{T}$ is triangularly angle rigid if and only if $\mathrm{Rank}(R_{\mathcal{A}}) = 2n-4$. Thus, the statement of this theorem is equivalent to that $\mathrm{Rank}(R^f_{\mathcal{A}}) = 2n-4$ if and only if $\mathrm{Rank}(R_{\mathcal{A}}) = 2n-4$.
Sufficiency: Since $R_{\mathcal{A}}p = 0$ by Lemma 1, one has

$$R_{\mathcal{A}}p = \begin{bmatrix} R^a_{\mathcal{A}} & R^f_{\mathcal{A}} \end{bmatrix}\begin{bmatrix} p_a \\ p_f \end{bmatrix} = R^a_{\mathcal{A}}p_a + R^f_{\mathcal{A}}p_f = 0. \tag{21}$$

Since $\mathrm{Rank}(R_{\mathcal{A}}) = 2n-4$, the four nonzero linearly independent vectors span the null space of $R_{\mathcal{A}}$. Then, $p = \beta_1 1_n \otimes [1,0]^\top + \beta_2 1_n \otimes [0,1]^\top + \beta_3\left(I_n \otimes \bar{R}(\pi/2)\right)p_0 + \beta_4 p_0$, where $\beta_i \in \mathbb{R}, i = 1,\ldots,4$, $p_0 \in \mathbb{R}^{2n}$ is an arbitrary generic realization of those angle constraints α^* among the sensor nodes. When $p_a \in \mathbb{R}^4$ is given, $\beta_i, i = 1,\ldots,4$ can be uniquely determined, under which p_f is then uniquely determined. According to (21), if $R^f_{\mathcal{A}}p_f = -R^a_{\mathcal{A}}p_a$ has a unique solution for p_f, then $\mathrm{Rank}(R^f_{\mathcal{A}}) = 2n-4$ because $R^f_{\mathcal{A}} \in \mathbb{R}^{2m(\mathcal{T})\times(2n-4)}$.
Necessity: When $\mathrm{Rank}(R^f_{\mathcal{A}}) = 2n-4$, the $(2n-4)$ columns in $R^f_{\mathcal{A}}$ are independent. Since $R^f_{\mathcal{A}}$ is a sub-matrix of $R_{\mathcal{A}}$ with the same number of rows, there exist at least $(2n-4)$ independent columns in $R_{\mathcal{A}}$, i.e., $\mathrm{Rank}(R_{\mathcal{A}}) \ge 2n-4$. By Theorem 1, one has that $\mathrm{Rank}(R_{\mathcal{A}}) \le 2n-4$. Therefore, $\mathrm{Rank}(R_{\mathcal{A}}) = 2n-4$. □

Appendix D. Proof of Lemma 7

We prove this lemma by contradiction. Obviously, it is impossible that a triangularly angle rigid trigraph $\mathcal{T}$ has one vertex without involving in any triangles in $\mathcal{A}$. Assume on the contrary that each vertex in $\mathcal{V}$ is associated with at least three triangles in $\mathcal{A}$. For the triangle $\triangle ijk$ in $\mathcal{A}$, each vertex of i,j,k will show thrice in $\mathcal{A}$ because $\{(i,j,k),(j,k,i),(j,i,k)\} \subseteq \mathcal{A}$. If one vertex is associated with 3 triangles in $\mathcal{A}$, it will show 9 times in $\mathcal{A}$. Therefore, if each vertex of $\mathcal{V}$ is associated with at least three triangles in $\mathcal{A}$, then the total shown times of the vertices of $\mathcal{V}$ in $\mathcal{A}$ should be at least $9n$. However, a minimally and triangularly angle rigid trigraph $\mathcal{T}$ only has $3*3*(n-2)$ places in $\mathcal{A}$ for all the vertices, which implies a contradiction with the assumption because $9n > 9(n-2)$. Therefore, there must be at least one vertex associated with only one or two triangles in $\mathcal{A}$. □

Appendix E. Proof of Theorem 7

To prove the necessity, we need to prove that if $\mathcal{T}'(\mathcal{V},\mathcal{A}')$ with $m(\mathcal{T}') = |\mathcal{V}|-2$ is triangularly angle rigid, then any triangular subtrigraph $\mathcal{T}''(\mathcal{V}'',\mathcal{A}'')$ of $\mathcal{T}'$ satisfies $m(\mathcal{T}'') \le |\mathcal{V}''|-2$. We prove this by contradiction. Suppose that there exists a subtrigraph $\mathcal{T}''$ of $\mathcal{T}'$ with $m(\mathcal{T}'') > |\mathcal{V}''|-2$. Let $R_{\mathcal{A}''} \in \mathbb{R}^{2m(\mathcal{T}'')\times 2|\mathcal{V}''|}$ be the triangular angle rigidity matrix of $\mathcal{T}''$. According to Theorem 1, one has $\mathrm{Rank}(R_{\mathcal{A}''}) \le 2|\mathcal{V}''|-4$ which implies that there are row dependences in the matrix $R_{\mathcal{A}''}$. Note that $[R_{\mathcal{A}''} \quad 0_{2m(\mathcal{T}'')\times 2(|\mathcal{V}|-|\mathcal{V}''|)}] \in \mathbb{R}^{2m(\mathcal{T}'')\times 2|\mathcal{V}|}$ is a submatrix of $R_{\mathcal{A}'} \in \mathbb{R}^{(2|\mathcal{V}|-4)\times 2|\mathcal{V}|}$ with the same number of columns. Then, the row dependences in $R_{\mathcal{A}''}$ imply row dependences in $R_{\mathcal{A}'}$. However, $\mathcal{T}'$ is minimally and triangularly angle rigid, and thus no row dependences should exist in the matrix $R_{\mathcal{A}'}$. This contradiction proves that $m(\mathcal{T}'') \le |\mathcal{V}''|-2$.

To prove the sufficiency, we need to prove that if $\mathcal{T}'(\mathcal{V},\mathcal{A}')$ is a L-trigraph, then $\mathcal{T}'$ is triangularly angle rigid. We prove the sufficiency by sequentially removing the nodes in $\mathcal{V}$ and their associated triangles in $\mathcal{T}'$ until the trigraph ends up with a single triangle that is itself triangularly angle rigid. This inductive proof works only when the following two propositions can be guaranteed at each step (Connelly & Guest, 2015, Thm 7.5.3). The first is that deleting a selected node from the trigraph $\mathcal{T}'$ will not change the triangle count condition $m(\mathcal{T}'') \le |\mathcal{V}''|-2$ on any subtrigraphs $\mathcal{T}''$ of $\mathcal{T}'$. The second is that if the trigraph after the deletion of a selected node is triangularly angle rigid, then the trigraph before this deletion is triangularly angle rigid. These indicate the importance of the selection of the node that will be deleted at each step. According to Lemma 7, there must be at least one node, which we label by $j \in \mathcal{V}$ that is only associated with 1 or 2 triangles in $\mathcal{T}'$. Therefore, we only need to check whether the two propositions hold when the node j and its associated triangles are deleted from $\mathcal{T}'$. The following three cases exist for the deletion of the node j's associated triangles.

(a) Type-I triangle deletion: If j is only associated with one triangle[3] $\triangle i_1i_2j$ in $\mathcal{T}'(\mathcal{V},\mathcal{A}')$, we now delete j and the associated triangle $\triangle i_1i_2j$ in $\mathcal{T}'$ to get the subtrigraph $\mathcal{T}_0(\mathcal{V}_0,\mathcal{A}_0)$ with $\mathcal{V}_0 = \mathcal{V}-\{j\}$ and $\mathcal{A}_0 = \mathcal{A}'-\{(j,i_1,i_2),(i_1,j,i_2),(i_1,i_2,j)\}$ (see Fig. 6). Note that the triangle count condition in any subtrigraphs of $\mathcal{T}_0$ is unchanged in comparison with that of $\mathcal{T}'$. Then, we prove that if $\mathcal{T}_0$ is triangularly angle rigid, then $\mathcal{T}'$ is triangularly angle rigid. The triangular angle rigidity matrix of $\mathcal{T}'$ can be written by $R_{\mathcal{A}'} = \begin{bmatrix} R_{\mathcal{A}_0} & 0 \\ \star & A_j^{\triangle i_1i_2j} \end{bmatrix}$, where $A_j^{\triangle i_1i_2j} = -\sin\alpha^*_{i_2ji_1}I_2 \ne 0$ under generic p, and $\star$ represents a matrix that will not affect the analysis. Then, one has $\mathrm{Rank}(R_{\mathcal{A}'}) = \mathrm{Rank}(R_{\mathcal{A}_0}) + \mathrm{Rank}(A_j^{\triangle i_1i_2j}) = \mathrm{Rank}(R_{\mathcal{A}_0}) + 2$, which implies that $\mathcal{T}_0$ is triangularly angle rigid if and only if $\mathcal{T}'$ is triangularly angle rigid.

(b) Type-II triangle deletion: If j is associated with two triangles $\triangle ji_1i_2$, $\triangle ji_2i_3$ and three vertices i_1,i_2,i_3 in $\mathcal{T}'(\mathcal{V},\mathcal{A}')$, we now delete j, $\triangle ji_1i_2$, $\triangle ji_2i_3$ from $\mathcal{T}'$ and add $\triangle i_1i_2i_3$ into $\mathcal{T}'$ to get the new trigraph $\mathcal{T}_0(\mathcal{V}_0,\mathcal{A}_0)$ with $\mathcal{V}_0 = \mathcal{V}-\{j\}$, $\mathcal{A}_0 = \mathcal{A}' - \mathcal{A}_1 - \mathcal{A}_2 + \mathcal{A}_3$, where $\mathcal{A}_1 = \{(j,i_1,i_2),(i_1,j,i_2),(i_1,i_2,j)\}$, $\mathcal{A}_2 =$

[3] The sequence of i_1, i_2, j in $\triangle i_1i_2j$ makes no difference.

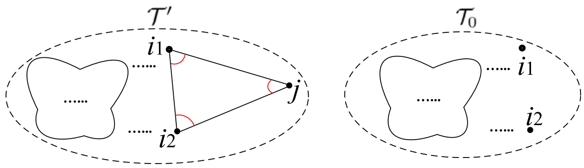


Fig. 6. Type-I triangle deletion: Node j is only associated with one triangle $\triangle i_1 i_2 j$ and two vertices in $\mathcal{T}'$.

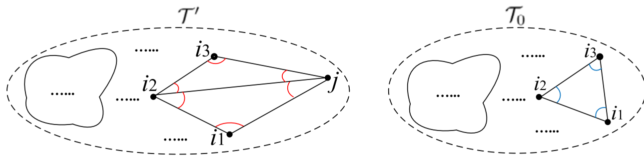


Fig. 7. Type-II triangle deletion: Node j is associated with two triangles $\triangle j i_1 i_2$, $\triangle j i_2 i_3$ and three vertices in $\mathcal{T}'$.

$\{(j, i_3, i_2),\ (i_3, j, i_2),\ (i_3, i_2, j)\}$, and $\mathcal{A}_3 = \{(i_3, i_1, i_2), (i_1, i_3, i_2), (i_1, i_2, i_3)\}$ (see Fig. 7). To proceed the proof, three parts need to prove: $\triangle i_1 i_2 i_3$ is addable (i.e., $(i_1, i_2, i_3) \notin \mathcal{A}'$); the triangle count condition holds for any subtrigraphs of $\mathcal{T}_0$; and angle rigid $\mathcal{T}_0 \Rightarrow$ angle rigid $\mathcal{T}'$.

To prove the first part, suppose on the contrary $(i_1, i_2, i_3) \in \mathcal{A}'$. Then, for $\mathcal{T}'$'s subtrigraph $\mathcal{T}_4(\mathcal{V}_4, \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)$ with $\mathcal{V}_4 = \{j, i_1, i_2, i_3\}$, one has $m(\mathcal{T}_4) = 3 > |\mathcal{V}_4| - 2 = 2$ which contradicts with the assumption of triangle count condition on subtrigraphs of $\mathcal{T}'$. Therefore, $(i_1, i_2, i_3) \notin \mathcal{A}'$.

To prove the second part, it is obvious that the triangle count condition still holds for $\mathcal{T}_0$'s subtrigraphs $\mathcal{T}''$ with $\{i_1, i_2, i_3\} \not\subseteq \mathcal{V}''$. To prove the remaining case, we consider an arbitrary subtrigraph $\mathcal{T}_5(\mathcal{V}_5, \mathcal{A}_5)$ of $\mathcal{T}'$ with $\{i_1, i_2, i_3\} \subseteq \mathcal{V}_5, j \notin \mathcal{V}_5$ and $m(\mathcal{T}_5) \le |\mathcal{V}_5| - 2$. Note that $\mathcal{A}_3 \nsubseteq \mathcal{A}_5$ since $\mathcal{A}_3 \nsubseteq \mathcal{A}'$ but $\mathcal{A}_3 \subseteq \mathcal{A}_0$. Therefore, we need to prove $m(\mathcal{T}_5) \le |\mathcal{V}_5| - 3$, otherwise the triangle count condition is violated in $\mathcal{T}_5$ after $\triangle i_1 i_2 i_3$ is added. Consider another new trigraph $\mathcal{T}_6(\mathcal{V}_5 \cup j, \mathcal{A}_5 \cup \mathcal{A}_1 \cup \mathcal{A}_2)$. Since $\mathcal{T}_6$ is a subtrigraph of $\mathcal{T}'$, one has $m(\mathcal{T}_6) = m(\mathcal{T}_5) + 2 \le |\mathcal{V}_6| - 2 = |\mathcal{V}_5| - 1$, i.e., $m(\mathcal{T}_5) \le |\mathcal{V}_5| - 3$.

To prove the third part, we only need to prove that row independence in $R_{\mathcal{A}_0}$ implies row independence in $R_{\mathcal{A}'}$. The triangular angle rigidity matrix of $\mathcal{T}_0$ is written by

$$R_{\mathcal{A}_0} = \begin{bmatrix} R_{\mathcal{A}''} & \star & \star & \star \\ 0 & A_{i_1}^{\triangle i_2 i_3 i_1} & A_{i_2}^{\triangle i_2 i_3 i_1} & A_{i_3}^{\triangle i_2 i_3 i_1} \end{bmatrix},$$

where $R_{\mathcal{A}_0} \in \mathbb{R}^{2m(\mathcal{T}_0) \times 2|\mathcal{V}_0|}$, and $R_{\mathcal{A}''} \in \mathbb{R}^{2(m(\mathcal{T}_0)-1) \times 2(|\mathcal{V}_0|-3)}$. The triangular angle rigidity matrix of $\mathcal{T}'$ is written by

$$R_{\mathcal{A}'} = \begin{bmatrix} R_{\mathcal{A}''} & \star & \star & \star & 0 \\ 0 & A_{i_1}^{\triangle i_2 j i_1} & A_{i_2}^{\triangle i_2 j i_1} & 0 & A_{j}^{\triangle i_2 j i_1} \\ 0 & 0 & A_{i_2}^{\triangle i_2 i_3 j} & A_{i_3}^{\triangle i_2 i_3 j} & A_{j}^{\triangle i_2 i_3 j} \end{bmatrix}.$$

Then, we construct the following rigidity matrix $P_1 \in \mathbb{R}^{6 \times 8}$ to describe the triangle constraints $\triangle i_2 i_3 i_1$, $\triangle i_2 j i_1$, $\triangle i_2 i_3 j$

$$P_1 = \begin{bmatrix} r_1^\top \\ r_2^\top \\ r_3^\top \\ r_4^\top \\ r_5^\top \\ r_6^\top \end{bmatrix} = \begin{bmatrix} A_{i_1}^{\triangle i_2 i_3 i_1} & A_{i_2}^{\triangle i_2 i_3 i_1} & A_{i_3}^{\triangle i_2 i_3 i_1} & 0 \\ A_{i_1}^{\triangle i_2 j i_1} & A_{i_2}^{\triangle i_2 j i_1} & 0 & A_{j}^{\triangle i_2 j i_1} \\ 0 & A_{i_2}^{\triangle i_2 i_3 j} & A_{i_3}^{\triangle i_2 i_3 j} & A_{j}^{\triangle i_2 i_3 j} \end{bmatrix},$$

where $r_i \in \mathbb{R}^{8 \times 1}, i = 1, \ldots, 6$ and $\mathrm{Rank}(P_1) = 4$ according to Theorem 4. Different from the proof of Laman's theorem (Laman, 1970; Whiteley, 1996, Thm 2.2.2), $R_{\mathcal{A}_0}$ and $R_{\mathcal{A}'}$ consist of matrix blocks instead of row vectors. To prove this part, it is equivalent to proving that row dependences in $R_{\mathcal{A}'}$ imply row dependences in $R_{\mathcal{A}_0}$. According to the definitions of $A_j^{\triangle i_2 j i_1}$ and $A_j^{\triangle i_2 i_3 j}$ in (6), the row dependences corresponding to vertex j in matrix $R_{\mathcal{A}'}$ imply that

$$\begin{cases} \omega_3 \cos\alpha^*_{i_1 i_2 j} - \omega_4 \sin\alpha^*_{i_1 i_2 j} - \omega_5 \frac{\sin\alpha^*_{i_3 j i_2}}{\sin\alpha^*_{i_2 j i_1}} = 0, \\ \omega_3 \sin\alpha^*_{i_1 i_2 j} + \omega_4 \cos\alpha^*_{i_1 i_2 j} - \omega_6 \frac{\sin\alpha^*_{i_3 j i_2}}{\sin\alpha^*_{i_2 j i_1}} = 0, \end{cases} \tag{22}$$

where $\omega_i \in \mathbb{R}, i = 3, 4, 5, 6$ are four scalars describing the row dependences of $r_3^\top, r_4^\top, r_5^\top, r_6^\top$ in matrix $R_{\mathcal{A}'}$. To prove the existence of row dependences in matrix $R_{\mathcal{A}_0}$, we only need to prove that r_1 and r_2 are linearly dependent to r_3, r_4, r_5, r_6 with the exactly same coefficients $\omega_3, \omega_4, \omega_5, \omega_6$, respectively, i.e.,

$$\begin{cases} \omega_1 r_1 + \omega_3 r_3 + \omega_4 r_4 + \omega_5 r_5 + \omega_6 r_6 = 0, \\ \omega_2 r_2 + \omega_3 r_3 + \omega_4 r_4 + \omega_5 r_5 + \omega_6 r_6 = 0, \end{cases} \tag{23}$$

where $\omega_1 \neq 0$ and $\omega_2 \neq 0$ are two scalars. Note that the fact $\mathrm{Rank}(P_1) = 4$ implies that r_1, r_2 are linearly dependent to r_3, r_4, r_5, r_6. However, if the coefficients in front of r_3, r_4, r_5, r_6 are not exactly $\omega_3, \omega_4, \omega_5, \omega_6$ in these dependences, one cannot directly conclude the existence of row dependences in $R_{\mathcal{A}_0}$. Therefore, we first calculate ω_1, ω_2 by using the row dependence corresponding to vertex i_1 in P_1, i.e.,

$$\begin{cases} \omega_1 \sin\alpha^*_{i_3 i_1 i_2} + \omega_3 \sin\alpha^*_{j i_1 i_2} = 0, \\ \omega_2 \sin\alpha^*_{i_3 i_1 i_2} + \omega_4 \sin\alpha^*_{j i_1 i_2} = 0. \end{cases} \tag{24}$$

Due to the fact that the sum of each row's elements is zero for triangular angle rigidity matrices, we do not need to check both the row dependences corresponding to vertices i_2 and i_3. More specifically, since $A_{i_2}^{\triangle i_2 i_3 i_1} = -A_{i_1}^{\triangle i_2 i_3 i_1} - A_{i_3}^{\triangle i_2 i_3 i_1}$, to prove (23), one only needs to verify the row dependence corresponding to vertex i_3 in P_1, which can be written by

$$\begin{cases} \omega_1 \cos\alpha^*_{i_1 i_2 i_3} - \omega_2 \sin\alpha^*_{i_1 i_2 i_3} \\ \quad = \frac{\sin\alpha^*_{i_2 i_3 j}}{\sin\alpha^*_{i_2 i_3 i_1}} (\omega_6 \sin\alpha^*_{j i_2 i_3} - \omega_5 \cos\alpha^*_{j i_2 i_3}), \\ \omega_1 \sin\alpha^*_{i_1 i_2 i_3} + \omega_2 \cos\alpha^*_{i_1 i_2 i_3} \\ \quad = -\frac{\sin\alpha^*_{i_2 i_3 j}}{\sin\alpha^*_{i_2 i_3 i_1}} (\omega_6 \cos\alpha^*_{j i_2 i_3} + \omega_5 \sin\alpha^*_{j i_2 i_3}). \end{cases} \tag{25}$$

By taking the first equation of (25) as an example, substituting (22), (24) into the first equation of (25) yields

$$\left[\frac{\gamma_1}{\sin\alpha^*_{i_3 j i_2}} - \frac{\cos\alpha^*_{i_1 i_2 i_3} \sin\alpha^*_{j i_1 i_2} \sin\alpha^*_{i_2 i_3 i_1}}{\sin\alpha^*_{i_2 j i_1} \sin\alpha^*_{i_3 i_1 i_2}} \right] \omega_3 + \left[\frac{\gamma_2}{\sin\alpha^*_{i_3 j i_2}} - \frac{\sin\alpha^*_{i_1 i_2 i_3} \sin\alpha^*_{j i_1 i_2} \sin\alpha^*_{i_2 i_3 i_1}}{\sin\alpha^*_{i_2 j i_1} \sin\alpha^*_{i_3 i_1 i_2}} \right] \omega_4 = 0, \tag{26}$$

where $\gamma_1 = (\cos\alpha^*_{j i_2 i_3} \cos\alpha^*_{i_1 i_2 j} - \sin\alpha^*_{j i_2 i_3} \sin\alpha^*_{i_1 i_2 j}) \times \sin\alpha^*_{i_2 i_3 j} = \cos\alpha^*_{i_1 i_2 i_3} \sin\alpha^*_{i_2 i_3 j}$ and $\gamma_2 = (\cos\alpha^*_{j i_2 i_3} \times \sin\alpha^*_{i_1 i_2 j} + \sin\alpha^*_{j i_2 i_3} \times \cos\alpha^*_{i_1 i_2 j}) \sin\alpha^*_{i_2 i_3 j} = \sin\alpha^*_{i_1 i_2 i_3} \sin\alpha^*_{i_2 i_3 j}$. The coefficient in front of ω_3 satisfies $\frac{\gamma_1}{\sin\alpha^*_{i_3 j i_2}} - \frac{\cos\alpha^*_{i_1 i_2 i_3} \sin\alpha^*_{j i_1 i_2} \sin\alpha^*_{i_2 i_3 i_1}}{\sin\alpha^*_{i_2 j i_1} \sin\alpha^*_{i_3 i_1 i_2}} = \cos\alpha^*_{i_1 i_2 i_3} (\frac{l_{i_2 j}}{l_{i_2 i_3}} - \frac{l_{j i_2}}{l_{l_{i_2 i_1}}} \frac{l_{i_2 i_1}}{l_{i_3 i_2}}) = 0$, where we used the law of sines. By using similar calculations, the coefficient in front of ω_4 in (26) also equals zero. The same case applies for the second equation of (25). Therefore, no matter what ω_3, ω_4 are in (22), (25) always holds, which implies that (23) holds and row dependences exist in $R_{\mathcal{A}_0}$.

(c) Type-III triangle deletion: If j is associated with two triangles $\triangle j i_1 i_2$, $\triangle j i_3 i_4$ and four vertices i_1, i_2, i_3, i_4 in $\mathcal{T}'(\mathcal{V}, \mathcal{A}')$, we now remove j, $\triangle j i_1 i_2$, $\triangle j i_3 i_4$ and add one triangle of $\triangle i_1 i_2 i_3$,

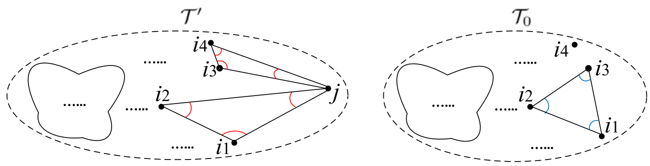


Fig. 8. Type-III triangle deletion: Node j is associated with two triangles $\triangle ji_1i_2$, $\triangle ji_3i_4$ and four vertices in $\mathcal{T}'$.

$\triangle i_1i_2i_4, \triangle i_1i_3i_4, \triangle i_2i_3i_4$ (without loss of genericity, consider that the added triangle is $\triangle i_1i_2i_3$ in the follow-up analysis) to get the new trigraph $\mathcal{T}_0(\mathcal{V}_0, \mathcal{A}_0)$ with $\mathcal{V}_0 = \mathcal{V} - \{j\}$ and $\mathcal{A}_0 = \mathcal{A}' - \mathcal{A}_1 - \mathcal{A}_6 + \mathcal{A}_3$, where $\mathcal{A}_1 = \{(j, i_1, i_2), (i_1, j, i_2), (i_1, i_2, j)\}$, $\mathcal{A}_6 = \{(j, i_3, i_4), (i_3, j, i_4), (i_3, i_4, j)\}$, $\mathcal{A}_3 = \{(i_3, i_1, i_2), (i_1, i_3, i_2), (i_1, i_2, i_3)\}$ (see Fig. 8). To proceed the proof, three parts need to prove: $\triangle i_1i_2i_3$ is addable; the triangle count condition holds for any subtrigraphs of $\mathcal{T}_0$; and angle rigid $\mathcal{T}_0 \Rightarrow$ angle rigid $\mathcal{T}'$.

To prove the first part, we consider $\mathcal{T}'$'s subtrigraph $\mathcal{T}_4(\mathcal{V}_4, \mathcal{A}_4)$ with $\mathcal{V}_4 = \{i_1, i_2, i_3, i_4\}$, $\mathcal{A}_4$ consisting of those triangles associated with the four vertices, and $m(\mathcal{T}_4) \le 4 - 2 = 2$. Then, we only need to prove $m(\mathcal{T}_4) \le 1$, i.e., one more triangle can be added into $\mathcal{T}_4$, which is one of $\triangle i_1i_2i_3, \triangle i_1i_2i_4, \triangle i_1i_3i_4, \triangle i_2i_3i_4$. Suppose on the contrary $m(\mathcal{T}_4) \ge 2$. Consider the subtrigraph $\mathcal{T}_5(\mathcal{V}_5, \mathcal{A}_5)$ of $\mathcal{T}'$ with $\mathcal{V}_5 = \{j, i_1, i_2, i_3, i_4\}$ and $\mathcal{A}_5 = \mathcal{A}_4 \cup \mathcal{A}_1 \cup \mathcal{A}_6$. It follows that $m(\mathcal{T}_5) = m(\mathcal{T}_4) + 2 \ge 4 > |\mathcal{V}_5| - 2$ which violates the triangle count condition on the subtrigraph of $\mathcal{T}'$. This contradiction implies $m(\mathcal{T}_4) \le 1$.

To prove the second part, we only need to prove that $\mathcal{T}'$'s subtrigraph $\mathcal{T}''(\mathcal{V}'', \mathcal{A}'')$ with $\{i_1, i_2, i_3, i_4\} \subseteq \mathcal{V}''$ and $j \notin \mathcal{V}''$ satisfies $m(\mathcal{T}'') \le |\mathcal{V}''| - 3$ since the triangle count condition on the other subtrigraphs of $\mathcal{T}'$ directly holds. Considering the subtrigraph $\mathcal{T}_7(\mathcal{V}'' \cup j, \mathcal{A}'' \cup \mathcal{A}_1 \cup \mathcal{A}_6)$ of $\mathcal{T}'$, one has $m(\mathcal{T}_7) = m(\mathcal{T}'') + 2 \le (|\mathcal{V}''| + 1) - 2$, which implies $m(\mathcal{T}'') \le |\mathcal{V}''| - 3$.

To prove the third part, we also aim to prove that row dependences in $R_{\mathcal{A}'}$ imply row dependences in $R_{\mathcal{A}_0}$. The triangular angle rigidity matrix of $\mathcal{T}_0$ can be written by

$$R_{\mathcal{A}_0} = \begin{bmatrix} R_{\mathcal{A}''} & \star & \star & \star & \star \\ 0 & A_{i_1}^{\triangle i_1i_2i_3} & A_{i_2}^{\triangle i_1i_2i_3} & A_{i_3}^{\triangle i_1i_2i_3} & 0 \end{bmatrix},$$

where $R_{\mathcal{A}''} \in \mathbb{R}^{2(n-4)\times 2(n-4)}$. The triangular angle rigidity matrix of $\mathcal{T}'$ is written by

$$R_{\mathcal{A}'} = \begin{bmatrix} R_{\mathcal{A}''} & \star & \star & \star & \star & 0 \\ 0 & A_{i_1}^{\triangle i_1ji_2} & A_{i_2}^{\triangle i_1ji_2} & 0 & 0 & A_j^{\triangle i_1ji_2} \\ 0 & 0 & 0 & A_{i_3}^{\triangle i_4i_3j} & A_{i_4}^{\triangle i_4i_3j} & A_j^{\triangle i_4i_3j} \end{bmatrix}.$$

Also, we construct the following rigidity matrix $P_2 \in \mathbb{R}^{(2M+2)\times 2|\mathcal{V}|}$ with $M = m(\mathcal{T}') = n - 2$ to describe all the triangle constraints in $\mathcal{T}'$ and $\triangle i_1i_2i_3$

$$\begin{aligned} P_2 &= \begin{bmatrix} r_1, & r_2, \dots & r_{2M-2}, & r_{2M-1}, & r_{2M}, & r_{2M+1}, & r_{2M+2} \end{bmatrix}^\top \\ &= \begin{bmatrix} R_{\mathcal{A}''} & \star & \star & \star & \star & 0 \\ 0 & A_{i_1}^{\triangle i_1ji_2} & A_{i_2}^{\triangle i_1ji_2} & 0 & 0 & A_j^{\triangle i_1ji_2} \\ 0 & 0 & 0 & A_{i_3}^{\triangle i_4i_3j} & A_{i_4}^{\triangle i_4i_3j} & A_j^{\triangle i_4i_3j} \\ 0 & A_{i_1}^{\triangle i_1i_2i_3} & A_{i_2}^{\triangle i_1i_2i_3} & A_{i_3}^{\triangle i_1i_2i_3} & 0 & 0 \end{bmatrix}, \end{aligned}$$

where $r_i \in \mathbb{R}^{2|\mathcal{V}|\times 1}$. We remark that the proof of this part is more challenging than the third part of Type-II triangle deletion since the last six rows of P_2 are independent. Thus, the proof of this part needs the involvement of all the triangle constrains in $\mathcal{T}'$ and all the vertices. After comparing the structure of $R_{\mathcal{A}_0}$, $R_{\mathcal{A}'}$, and P_2, the aim of this part is to prove that linear dependence of $\{r_1, r_2, \dots, r_{2M-1}, r_{2M}\}$ will imply linear dependence in $\{r_1, r_2, \dots, r_{2M-4}, r_{2M+1}, r_{2M+2}\}$. The row dependence in $R_{\mathcal{A}'}$ can be described by

$$\omega_1 r_1 + \omega_2 r_2 + \cdots + \omega_{2M-1} r_{2M-1} + \omega_{2M} r_{2M} = 0, \tag{27}$$

where $\omega_i, i = 1, \dots, 2M$ are scalars which are not all zeros. Using the definitions of $A_j^{\triangle i_1i_2j}, A_j^{\triangle i_4i_3j}$, the row dependence corresponding to vertex j in $R_{\mathcal{A}'}$ is written by

$$\begin{aligned} &\omega_{2M-3}\cos\alpha^*_{i_2i_1j} - \omega_{2M-2}\sin\alpha^*_{i_2i_1j} = \omega_{2M-1}\frac{\sin\alpha^*_{i_3ji_4}}{\sin\alpha^*_{i_1ji_2}}, \\ &\omega_{2M-3}\sin\alpha^*_{i_2i_1j} + \omega_{2M-2}\cos\alpha^*_{i_2i_1j} = \omega_{2M}\frac{\sin\alpha^*_{i_3ji_4}}{\sin\alpha^*_{i_1ji_2}}. \end{aligned} \tag{28}$$

The row dependence corresponding to vertex i_2 in $R_{\mathcal{A}'}$ is

$$\begin{aligned} &\omega_1 r_1(n_1) + \cdots + \omega_{2M-4} r_{2M-4}(n_1) = \omega_{2M-3}\sin\alpha^*_{ji_2i_1}, \\ &\omega_1 r_1(n_2) + \cdots + \omega_{2M-4} r_{2M-4}(n_2) = \omega_{2M-2}\sin\alpha^*_{ji_2i_1}, \end{aligned} \tag{29}$$

where $n_1 = 2|\mathcal{V}| - 7$, $n_2 = 2|\mathcal{V}| - 6$, and $r_i(n_1) \in \mathbb{R}$ denotes the n_1th element of the vector r_i. Since the last ten columns of P_2 are indexed by i_1, i_2, i_3, i_4, j, respectively, $r_1(n_1)$ and $r_1(n_2)$ are vector r_1's two elements corresponding to the vertex i_2. The row dependence corresponding to vertex i_3 in $R_{\mathcal{A}'}$ can be written by

$$\begin{aligned} &\omega_1 r_1(n_3) + \cdots + \omega_{2M-4} r_{2M-4}(n_3) + \omega_{2M-1}\sin\alpha^*_{i_4i_3j}\cos\alpha^*_{ji_4i_3} \\ &= \omega_{2M}\sin\alpha^*_{i_4i_3j}\sin\alpha^*_{ji_4i_3}, \\ &\omega_1 r_1(n_4) + \cdots + \omega_{2M-4} r_{2M-4}(n_4) + \omega_{2M-1}\sin\alpha^*_{i_4i_3j}\sin\alpha^*_{ji_4i_3} \\ &= -\omega_{2M}\sin\alpha^*_{i_4i_3j}\cos\alpha^*_{ji_4i_3}, \end{aligned} \tag{30}$$

where $n_3 = 2|\mathcal{V}| - 5$, and $n_4 = 2|\mathcal{V}| - 4$. Also, the row dependence corresponding to i_4 in $R_{\mathcal{A}'}$ can be written by

$$\begin{aligned} &\omega_1 r_1(n_5) + \cdots + \omega_{2M-4} r_{2M-4}(n_5) + \omega_{2M-1} A_{i_4}^{\triangle i_4i_3j}(1,1) \\ &\quad + \omega_{2M} A_{i_4}^{\triangle i_4i_3j}(2,1) = 0, \\ &\omega_1 r_1(n_6) + \cdots + \omega_{2M-4} r_{2M-4}(n_6) + \omega_{2M-1} A_{i_4}^{\triangle i_4i_3j}(1,2) \\ &\quad + \omega_{2M} A_{i_4}^{\triangle i_4i_3j}(2,2) = 0 \end{aligned} \tag{31}$$

where $n_5 = 2|\mathcal{V}| - 3$, $n_6 = 2|\mathcal{V}| - 2$, and $A_{i_4}^{\triangle i_4i_3j}(i,j)$ represents the element of the ith row and jth column of $A_{i_4}^{\triangle i_4i_3j}$. Now, we aim to prove the existence of row dependence in $R_{\mathcal{A}_0}$, i.e., in $\{r_1, r_2, \dots, r_{2M-4}, r_{2M+1}, r_{2M+2}\}$. We consider the first case where in the row dependence (27), the coefficients in front of r_{2M-1}, r_{2M} are zero, i.e., $\omega_{2M-1} = 0$ and $\omega_{2M} = 0$. Then, according to (28), one has $\omega_{2M-3} = 0$ and $\omega_{2M-2} = 0$ because $\bar{R}(\alpha_{i_2i_1j})$ is a rotation matrix. This indicates that the row dependences of $R_{\mathcal{A}'}$ must exist in its first row block, i.e., $[R_{\mathcal{A}''} \star \star \star \star]$, which is also the first row block of $R_{\mathcal{A}_0}$. Therefore, row dependences exist in $R_{\mathcal{A}_0}$. Then, we consider the remaining case where in the row dependence (27), the coefficients in front of r_{2M-1}, r_{2M} are not all zero, i.e., at least one of $\omega_{2M-1}, \omega_{2M}$ is nonzero. In this case, $A_{i_3}^{\triangle i_4i_3j}, A_{i_4}^{\triangle i_4i_3j}$ in $R_{\mathcal{A}'}$ are involved in the row dependence (27). According to (28), one has that at least one of $\omega_{2M-3}, \omega_{2M-2}$ is nonzero, i.e., $A_{i_1}^{\triangle i_1ji_2}, A_{i_2}^{\triangle i_1ji_2}$ in $R_{\mathcal{A}'}$ are also involved in the row dependence (27). Then, there must exist a subtrigraph $\mathcal{T}_7(\mathcal{V}_7, \mathcal{A}_7)$ of $\mathcal{T}'$ with $\{j, i_1, i_2, i_3, i_4\} \subseteq \mathcal{V}_7$, $\{\mathcal{A}_1, \mathcal{A}_6\} \subseteq \mathcal{A}_7$, and $m(\mathcal{T}_7) > |\mathcal{V}_7| - 2$, because the triangular angle rigidity matrix $R_{\mathcal{A}'}$ has row dependences. After the Type-III triangle deletion, $\mathcal{T}_0$ must have a subtrigraph $\mathcal{T}_8(\mathcal{V}_8, \mathcal{A}_8)$ with $\mathcal{V}_8 = \mathcal{V}_7 - \{j\}$ and $\mathcal{A}_8 = \mathcal{A}_7 - \mathcal{A}_1 - \mathcal{A}_6 + \mathcal{A}_3$, which implies $|\mathcal{V}_8| = |\mathcal{V}_7| - 1$ and $m(\mathcal{T}_8) = m(\mathcal{T}_7) - 1$. Since $m(\mathcal{T}_7) > |\mathcal{V}_7| - 2$, one has $m(\mathcal{T}_8) > |\mathcal{V}_8| - 2$, which implies that $\mathcal{T}_8$ is over-constrained, i.e., row dependences exist in $\mathcal{T}_8$. Since $\mathcal{T}_8$ is a subtrigraph of $\mathcal{T}_0$, row dependences exist in $R_{\mathcal{A}_0}$. Combining the above two cases yields that row dependences in $R_{\mathcal{A}'}$ implies row dependences in $R_{\mathcal{A}_0}$. According to Whiteley (1996), one has that independent $R_{\mathcal{A}_0} \Rightarrow$ independent $R_{\mathcal{A}'}$ for all generic configurations. Using Theorem 4, one has that triangularly angle rigid $\mathcal{T}_0 \Rightarrow$ triangularly angle rigid $\mathcal{T}'$. $\square$

Appendix F. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.automatica.2022.110414.

References

Aspnes, James, Eren, Tolga, Goldenberg, David Kiyoshi, Morse, A. Stephen, Whiteley, Walter, Yang, Yang Richard, et al. (2006). A theory of network localization. *IEEE Transactions on Mobile Computing*, *5*(12), 1663–1678.

Bishop, Adrian N., Anderson, Brian D. O., Fidan, Baris, Pathirana, Pubudu N., & Mao, Guoqiang (2009). Bearing-only localization using geometrically constrained optimization. *IEEE Transactions on Aerospace and Electronic Systems*, *45*(1), 308–320.

Cao, Kun, Han, Zhimin, Lin, Zhiyun, & Xie, Lihua (2021). Bearing-only distributed localization: A unified barycentric approach. *Automatica*, *133*, Article 109834.

Cao, Ming, Yu, Changbin, & Anderson, Brian D. O. (2011). Formation control using range-only measurements. *Automatica*, *47*(4), 776–781.

Chen, Liangming (2022). Some technical details of angle-based localization in 2D. Appendix F in https://doi.org/10.1016/j.automatica.2022.110414.

Chen, Liangming, Cao, Ming, & Li, Chuanjiang (2021). Angle rigidity and its usage to stabilize multi-agent formations in 2D. *IEEE Transactions on Automatic Control*, *66*(8), 3667–3681.

Cominelli, Marco, Patras, Paul, & Gringoli, Francesco (2019). Dead on arrival: An empirical study of the Bluetooth 5.1 positioning system. In *Proceedings of the 13th international workshop on wireless network testbeds, experimental evaluation & characterization* (pp. 13–20).

Connelly, R., & Guest, S. D. (2015). *Frameworks, tensegrities and symmetry: understanding stable structures*. Cornell Univ., College Arts Sci..

Diao, Yingfei, Lin, Zhiyun, & Fu, Minyue (2014). A barycentric coordinate based distributed localization algorithm for sensor networks. *IEEE Transactions on Signal Processing*, *62*(18), 4760–4771.

Eren, Tolga, Goldenberg, O. K., Whiteley, Walter, Yang, Yang Richard, Morse, A. Stephen, Anderson, Brian D. O., et al. (2004). Rigidity, computation, and randomization in network localization. *4*, In *IEEE INFOCOM 2004* (pp. 2673–2684). IEEE.

Eren, Tolga, Whiteley, Walter, & Belhumeur, Peter N. (2006). Using angle of arrival (bearing) information in network localization. In *Proceedings of the 45th IEEE conference on decision and control* (pp. 4676–4681). IEEE.

Fang, Xu, Li, Xiaolei, & Xie, Lihua (2020). Angle-displacement rigidity theory with application to distributed network localization. *IEEE Transactions on Automatic Control*, *66*(6), 2574–2587.

Han, Tingrui, Lin, Zhiyun, Zheng, Ronghao, Han, Zhimin, & Zhang, Haitao (2017). A barycentric coordinate based approach to three-dimensional distributed localization for wireless sensor networks. In *2017 13th IEEE international conference on control & automation (ICCA)* (pp. 600–605). IEEE.

Henneberg, Lebrecht (1911). *Die graphische statik der starren systeme, Vol. 31*. BG Teubner.

Jiang, Bomin, Anderson, Brian D. O., & Hmam, Hatem (2019). 3-D relative localization of mobile systems using distance-only measurements via semidefinite optimization. *IEEE Transactions on Aerospace and Electronic Systems*, *56*(3), 1903–1916.

Jiang, Bomin, Deghat, Mohammad, & Anderson, Brian D. O. (2016). Simultaneous velocity and position estimation via distance-only measurements with application to multi-agent system control. *IEEE Transactions on Automatic Control*, *62*(2), 869–875.

Jing, Gangshan, Wan, Changhuang, & Dai, Ran (2022). Angle-based sensor network localization. *IEEE Transactions on Automatic Control*, *67*(2), 840–855.

Laman, Gerard (1970). On graphs and rigidity of plane skeletal structures. *Journal of Engineering Mathematics*, *4*(4), 331–340.

Li, Xiaolei, Luo, Xiaoyuan, & Zhao, Shiyu (2019). Globally convergent distributed network localization using locally measured bearings. *IEEE Transactions on Control of Network Systems*, *7*(1), 245–253.

Lin, Zhiyun, Fu, Minyue, & Diao, Yingfei (2015). Distributed self localization for relative position sensing networks in 2D space. *IEEE Transactions on Signal Processing*, *63*(14), 3751–3761.

Lin, Zhiyun, Han, Tingrui, Zheng, Ronghao, & Fu, Minyue (2016). Distributed localization for 2-D sensor networks with bearing-only measurements under switching topologies. *IEEE Transactions on Signal Processing*, *64*(23), 6345–6359.

Mao, Guoqiang, Fidan, Barış, & Anderson, Brian D. O. (2007). Wireless sensor network localization techniques. *Computer Networks*, *51*(10), 2529–2553.

Nguyen, Thien-Minh, Qiu, Zhirong, Nguyen, Thien Hoang, Cao, Muqing, & Xie, Lihua (2019). Persistently excited adaptive relative localization and time-varying formation of robot swarms. *IEEE Transactions on Robotics*, *36*(2), 553–560.

Shames, Iman, Bishop, Adrian N., & Anderson, Brian D. O. (2012). Analysis of noisy bearing-only network localization. *IEEE Transactions on Automatic Control*, *58*(1), 247–252.

Trinh, Minh Hoang, Lee, Byung-Hun, Ye, Mengbin, & Ahn, Hyo-Sung (2018). Bearing-based formation control and network localization via global orientation estimation. In *2018 IEEE conference on control technology and applications (CCTA)* (pp. 1084–1089). IEEE.

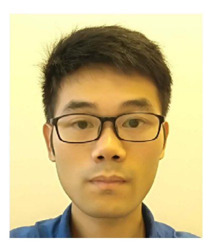
Whiteley, Walter (1996). Some matroids from discrete applied geometry. *Contemporary Mathematics*, *197*, 171–312.

Ye, Mengbin, Anderson, Brian D. O., & Yu, Changbin (2017). Bearing-only measurement self-localization, velocity consensus and formation control. *IEEE Transactions on Aerospace and Electronic Systems*, *53*(2), 575–586.

Zhao, Shiyu, & Zelazo, Daniel (2016a). Bearing rigidity and almost global bearing-only formation stabilization. *IEEE Transactions on Automatic Control*, *61*(5), 1255–1268.

Zhao, Shiyu, & Zelazo, Daniel (2016b). Localizability and distributed protocols for bearing-based network localization in arbitrary dimensions. *Automatica*, *69*, 334–341.

Zhao, Shiyu, & Zelazo, Daniel (2019). Bearing rigidity theory and its applications for control and estimation of network systems: Life beyond distance rigidity. *IEEE Control Systems Magazine*, *39*(2), 66–83.



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