Automatica 146 (2022) 110605

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Simultaneous localization and formation using angle-only measurements in 2D[☆]

Liangming Chen[a,b], Lihua Xie[b], Xiaolei Li[c], Xu Fang[b], Mir Feroskhan[a,*]

[a] School of Mechanical and Aerospace Engineering, Nanyang Technological University, Singapore
[b] School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore
[c] School of Electrical Engineering, Yanshan University, China

ARTICLE INFO

Article history:
Received 6 August 2021
Received in revised form 2 May 2022
Accepted 20 July 2022
Available online 29 September 2022

Keywords:
Multi-agent system
Simultaneous localization and formation
Angle measurements
Formation control
Distributed localization

ABSTRACT

This paper solves the simultaneous localization and formation (SLAF) problem for a multi-agent system moving in 2D plane. The multi-agent system consists of leaders who have the knowledge of their absolute positions in the global coordinate frame, and followers who do not know their absolute positions but have angle-only measurements and communication with respect to their neighboring agents. The aim of SLAF is to simultaneously localize and control the followers such that a desired formation among the leaders and followers can be achieved by using locally available sensing and communication information. To handle the challenging situation where the formation becomes unlocalizable at some nongeneric configurations, a perturbation-based SLAF algorithm is proposed such that the SLAF task can be achieved with an asymptotic convergence. To meet different tasks' requirements, three types of distributed SLAF algorithms are designed for the followers when the leaders are static, move with constant, or time-varying velocities, respectively. The effect of measurement noises, extension to other types of sensor measurements, requirement on agents' coordinate frames, collision and collinearity avoidance are also discussed. To validate the theoretical results, simulation examples corresponding to the discussed scenarios are carried out.

1. Introduction

Motivated by broad applications in multi-robot search and rescue (Shiroma, Chiu, Sato, & Matsuno, 2005), navigation and autonomy (Ravankar et al., 2018), and swarm intelligence and space exploration (Hu, Niu, Carrasco, Lennox, & Arvin, 2020), two fundamental problems have been extensively investigated recently in the research community of networked multi-agent systems, namely network localization and formation shape control (Han, Guo, Xie, & Lin, 2018). In the network localization problem, the multi-agent system consists of leaders who have the knowledge of their absolute positions in the global coordinate frame, and followers who do not know their absolute positions but have sensor measurements with respect to their neighbors and wireless channels to communicate with their neighbors (Lin, Han, Zheng, & Yu, 2017; Shames, Bishop, & Anderson, 2012). The aim of the network localization problem is to determine the positions of the followers by using their available communication and measurement information (Zhao & Zelazo, 2016). According to the categories of sensor measurements, network localization can be mainly classified into relative position, distance, bearing, and angle-based localization (Chen, 2022; Diao, Lin, & Fu, 2014; Fang, Li, & Xie, 2021; Jing, Wan, & Dai, 2021; Lin et al., 2017; Shames et al., 2012; Zhao & Zelazo, 2016), to name a few. Differently, the aim of formation shape control is to achieve a prescribed geometric shape among the agents by using inter-agent communication and measurements, such as absolute positions, relative positions, distances, bearings and angles (Ahn, 2020; Anderson, Yu, Fidan, & Hendrickx, 2008; Chen, Cao, & Li, 2021; Jing, Zhang, Lee, & Wang, 2019; Oh & Ahn, 2013; Oh, Park, & Ahn, 2015; Zhao & Zelazo, 2016).

Since the configuration of a mobile multi-agent network is dynamic and each agent usually has limited sensing capability, the localization and formation control of those networked agents are usually required to be achieved simultaneously in practical tasks. This motivates the study of simultaneous localization and formation which aims to simultaneously localize and control the followers such that a desired formation among leaders and followers can be achieved. The available information of each

☆ The work of Xiaolei Li was supported in part by the National Natural Science Foundation of China under Grants 61903319 and 62103352. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Dimitra Panagou under the direction of Editor Christos G. Cassandras.

* Corresponding author.
E-mail addresses: liangming.chen@ntu.edu.sg (L. Chen), ELHXIE@ntu.edu.sg (L. Xie), xiaolei@ysu.edu.cn (X. Li), fa0001xu@e.ntu.edu.sg (X. Fang), mir.feroskhan@ntu.edu.sg (M. Feroskhan).

https://doi.org/10.1016/j.automatica.2022.110605
0005-1098/

follower in the SLAF problem consists of communication information from its neighbors and measurement information such as distances, bearings, relative positions, and angles, with respect to its neighbors. Recently, some efforts have been made on simultaneous *relative* localization and formation problem, where relative positions among neighboring agents are localized using the measurements of bearing (Ye, Anderson, & Yu, 2017), distance and relative velocity (Han et al., 2018), or distance and self-displacement (Guo, Li, & Xie, 2019; Nguyen, Qiu, Nguyen, Cao, & Xie, 2019). Although relative localization is adequate for the achievement of some formation tasks, the awareness of an agent's absolute position in the global coordinate frame is also important, especially in the execution of practical tasks, such as search and rescue. Motivated by this, simultaneous localization and formation algorithms have been developed in Guo, Jayawardhana, Lee, and Shim (2020) and Huang, Farritor, Qadi, and Goddard (2006) which rely on inter-agent relative position measurements or the knowledge of the global coordinate frame at the initial time.

Unlike the previous works on localization of static networks (Chen, 2022; Diao et al., 2014; Fang et al., 2021; Jing et al., 2021; Lin, Han, & Cao, 2020; Lin, Han, Zheng, & Fu, 2016; Zhao & Zelazo, 2016), formation control (Chen et al., 2021; Hu et al., 2020; Jing et al., 2019; Zhao & Zelazo, 2016), and simultaneous relative localization and formation (Guo et al., 2020, 2019; Han et al., 2018; Nguyen et al., 2019), we investigate the simultaneous localization and formation problem where the followers' absolute positions are estimated using their *angle-only* measurements and communication information with respect to their neighbors. Note that angle measurements are more accessible than relative position measurements, but their nonlinearity makes the estimator and control design more challenging. Specifically, the multi-agent network is unlocalizable when they reach a nongeneric configuration, such as collinear configuration. In this paper, to meet different demands during the execution of practical tasks, three types of SLAF algorithms are designed for the followers when the leaders are static, move with constant velocities, or move with time-varying velocities, respectively. The main advantages of the proposed SLAF approach lie in three aspects. Firstly, the proposed SLAF algorithms allow the followers to have angle-only measurements which can be acquired from monocular cameras and directional sensor arrays. Secondly, the leaders in the SLAF problem can be static, move with constant or time-varying velocities. For the case of leaders moving with constant velocities, to reduce communication burden, the estimation of the constant velocity is avoided in the followers' SLAF algorithm. Thirdly, to deal with unlocalizable situations when the multi-agent network reaches nongeneric configurations, a perturbation-based algorithm is proposed to achieve the SLAF task.

The rest of the paper is organized as follows. Section 2 presents the preliminaries on angle measurements and angle-based localization. Section 3 discusses SLAF for the case of static leaders. Sections 4 and 5 discuss SLAF for the cases of dynamic leaders moving with constant and time-varying velocities, respectively. Some further discussion is conducted in Section 6. Simulations are provided in Section 7.

2. Preliminaries on angle measurements and angle-based localization

2.1. Notations

Consider a 2D multi-agent network consisting of $n_l \geq 2$ leaders and $n_f > 0$ followers. Let $\mathcal{V}_l = \{1, 2, \ldots, n_l\}$ be the set of leaders, whose positions, denoted by $p_l = [p_1^\top, p_2^\top, \ldots, p_{n_l}^\top]^\top \in \mathbb{R}^{2n_l}$, are known by themselves. Let $\mathcal{V}_f = \{n_l + 1, n_l + 2, \ldots, n\}$ be the set of followers with $n_f + n_l = n$, whose positions, denoted by $p_f = [p_{n_l+1}^\top, p_{n_l+2}^\top, \ldots, p_n^\top]^\top \in \mathbb{R}^{2n_f}$, are unknown. We assume that no overlapping points exist in $p = [p_l^\top, p_f^\top]^\top$. Let I_2, 1_n, $\otimes$, $\lambda_{\max}$, $\lambda_{\min}$, det() be the 2-by-2 identity matrix, $n \times 1$ column vector of all ones, the Kronecker product, the maximum eigenvalue, the minimum eigenvalue of a symmetric matrix, and the determinant of a square matrix, respectively. Denote by $\bar{R}(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO(2)$ the 2D rotation matrix with rotation angle $\theta \in [0, 2\pi)$. To avoid confusion, the notation of a quantity, e.g., x, represents that it is constant, and $x(t)$ represents that it is time-varying, and x^* represents that it is desired. For $x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, let $x^\beta = [x_1^\beta, \ldots, x_n^\beta]^\top$ and $\text{sig}^\beta(x) = [\text{sig}^\beta(x_1), \ldots, \text{sig}^\beta(x_n)]^\top$ where $\text{sig}^\beta(x_i) = \text{sgn}(x_i)|x_i|^\beta$ and $\text{sgn}(\cdot)$ represents the signum function.

2.2. Angle measurements and angle-induced linear equations

We introduce angle measurements and their angle-induced linear equations based on Chen (2022) and Chen et al. (2021). Note that describing agents' geometric relationship as a linear algebraic equation is an efficient way to transfer a nonlinear localization problem into a linear least-square optimization problem (Lin et al., 2016; Zhao & Zelazo, 2016).

Similar to Chen et al. (2021), we define the signed interior angle measurement $\alpha_{kij} \in [0, 2\pi)$ among agents k, i, j as

$$\alpha_{kij} := \begin{cases} \arccos(b_{ij}^\top b_{ik}), & \text{if } b_{ij}^\top b_{ik}^\perp \geq 0, \\ 2\pi - \arccos(b_{ij}^\top b_{ik}), & \text{otherwise}, \end{cases} \tag{1}$$

where $b_{ij} := \frac{p_j - p_i}{\|p_j - p_i\|}$ is the bearing from agent i to agent j, $p_i \neq p_j$, $i, j \in \mathcal{V}_l \cup \mathcal{V}_f$, $b_{ik}^\perp := \bar{R}(\frac{\pi}{2}) b_{ik} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] b_{ik}$. Note that α_{kij} represents the angle rotating from the bearing direction b_{ik} to the bearing direction b_{ij} under the counterclockwise direction. In $\triangle ijk$, one has

$$\frac{p_j - p_i}{\|p_j - p_i\|} = \bar{R}(\alpha_{kij}) \frac{p_k - p_i}{\|p_k - p_i\|}. \tag{2}$$

Using the law of sines $\frac{\|p_k - p_i\|}{\|p_j - p_i\|} = \frac{\sin\alpha_{ijk}}{\sin\alpha_{jki}}$ and the fact (2), the angle-induced linear equation (Chen, 2022) in $\triangle ijk$ can be written as

$$f_{ijk}(\alpha, p) = A_i^{\triangle ijk}(\alpha) p_i + A_j^{\triangle ijk}(\alpha) p_j + A_k^{\triangle ijk}(\alpha) p_k = 0, \tag{3}$$

where the coefficient matrices

$$A_i^{\triangle ijk}(\alpha) := \left(\sin\alpha_{jki} \bar{R}(\alpha_{kij}) - \sin\alpha_{ijk} I_2\right) \in \mathbb{R}^{2\times 2},$$
$$A_j^{\triangle ijk}(\alpha) := \sin\alpha_{ijk} I_2 \in \mathbb{R}^{2\times 2},$$
$$A_k^{\triangle ijk}(\alpha) := -\sin\alpha_{jki} \bar{R}(\alpha_{kij}) \in \mathbb{R}^{2\times 2}$$

are only related to the measured interior angles $\alpha_{jki}, \alpha_{ijk}, \alpha_{kij}$. When p_i, p_j, p_k are collinear, the linear Eq. (3) is degraded and cannot play any role. According to (1), angle measurements are independent of the orientations of agents' coordinate frames, which is an advantage over relative position or bearing measurements.

Remark 1. For a static network with randomly chosen configuration p, the probability of the existence of collinear points in p is zero (Connelly & Guest, 2015, Thm 7.2.1). However, for dynamic networks, particularly the multi-agent network conducting the SLAF task, the chance of collinearity among neighboring agents is inevitable. Therefore, we will pay special attention to collinear configurations in the follow-up analysis since they will make the linear Eq. (3) invalid.

We now introduce the notion of *triangular angularity* to describe a multi-agent network with multiple triangles and their associated angles. For the vertex set $\mathcal{V} = \{1, 2, \dots, n\}$ where node $i \in \mathcal{V}$ corresponds to agent i, define a three-vertex *triplet* (i, j, k) to describe the angle α_{ijk}. Then, we define $\mathcal{A} \subset \mathcal{V} \times \mathcal{V} \times \mathcal{V} = \{(i, j, k), i, j, k \in \mathcal{V}, i \neq j \neq k\}$ as an angle set, of which each element is a triplet. Since $\alpha_{ijk} + \alpha_{kji} \equiv 2\pi$, constraining α_{ijk} is equivalent to constraining α_{kji}. Thus (i, j, k) and (k, j, i) are interchangeable with each other in $\mathcal{A}$. Then, the combination of the vertex set $\mathcal{V}$, the angle set $\mathcal{A}$ and the position configuration $p \in \mathbb{R}^{2n}$ is called an *angularity* which we denote by $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$(Chen et al., 2021). We say $\mathcal{A}$ is a *triangular angle set* if for every $(i_1, j_1, k_1) \in \mathcal{A}$, there also exists $\{(j_1, k_1, i_1), (k_1, i_1, j_1)\} \subset \mathcal{A}$. The number of triangles in a triangular angle set $\mathcal{A}$ is denoted by $m \in \mathbb{N}^+$. We say $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is a *triangular angularity* if $\mathcal{A}$ is a triangular angle set (Chen, 2022). If $(i, j, k) \in \mathcal{A}$, then $\{j, k\} \in \mathcal{N}_i$, $\{i, k\} \in \mathcal{N}_j$, $\{i, j\} \in \mathcal{N}_k$ where $\mathcal{N}_i$ represents i's neighbor set.

Writing all the angle-induced linear Eqs. (3) from a triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ into a compact form, one has $R_{\mathcal{A}}(\alpha)p = 0$ where $R_{\mathcal{A}}(\alpha) \in \mathbb{R}^{2m\times 2n}$ is defined as the *angle measurement matrix* which can be written as

$$\begin{array}{c} \\ \text{1st } \triangle \\ \cdots \\ \triangle ijk \\ \cdots \\ m\text{th } \triangle \end{array}
\begin{array}{c} \begin{array}{ccccccc} \cdots & \text{Vertex } i & \cdots & \text{Vertex } j & \cdots & \text{Vertex } k & \cdots \end{array} \\
\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & A_i^{\triangle ijk} & 0 & A_j^{\triangle ijk} & 0 & A_k^{\triangle ijk} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array} \tag{4}$$

whose row blocks are indexed by the triangles in $\mathcal{A}$ and column blocks the vertices in $\mathcal{V}$. According to Chen (2022), Chen et al. (2021, Lem2), the maximum rank of $R_{\mathcal{A}}(\alpha)$ is $2n - 4$ since the kernel of $R_{\mathcal{A}}(\alpha)$ always includes translation, rotation and scaling of p. Although the angle measurement matrix is denoted as $R_{\mathcal{A}}(\alpha)$, it is a function of $\sin(\alpha_{jik}(p))$ and $\cos(\alpha_{jik}(p))$, $(j, i, k) \in \mathcal{A}$.

2.3. Localizability conditions and localization law

For a static multi-agent network, the aim of localization can be formulated as a least-square optimization problem with cost function (Fang et al., 2021; Zhao & Zelazo, 2016)

$$J(\hat{p}) = \hat{p}^\top R_{\mathcal{A}}^\top(\alpha) R_{\mathcal{A}}(\alpha)\hat{p} \tag{5}$$

where $\hat{p} = [p_l^\top, \hat{p}_f^\top]^\top \in \mathbb{R}^{2n}$ and $\hat{p}_f \in \mathbb{R}^{2n_f}$ denotes the estimates of the followers' positions p_f. If we partition the angle measurement matrix $R_{\mathcal{A}}(\alpha) = [R_{\mathcal{A}}^l \; R_{\mathcal{A}}^f]$ into leaders' part $R_{\mathcal{A}}^l \in \mathbb{R}^{2m\times 2n_l}$ and followers' part $R_{\mathcal{A}}^f \in \mathbb{R}^{2m\times 2n_f}$, we define the matrix $\mathcal{L}(\alpha)$ in the form of

$$\mathcal{L}(\alpha) = R_{\mathcal{A}}^\top(\alpha) R_{\mathcal{A}}(\alpha) = \begin{bmatrix} \mathcal{L}_{ll} & \mathcal{L}_{lf} \\ \mathcal{L}_{fl} & \mathcal{L}_{ff} \end{bmatrix}, \tag{6}$$

where $\mathcal{L}_{ll} = (R_{\mathcal{A}}^l)^\top R_{\mathcal{A}}^l \in \mathbb{R}^{2n_l\times 2n_l}$, $\mathcal{L}_{lf} = (R_{\mathcal{A}}^l)^\top R_{\mathcal{A}}^f \in \mathbb{R}^{2n_l\times 2n_f}$, $\mathcal{L}_{fl} = (R_{\mathcal{A}}^f)^\top R_{\mathcal{A}}^l \in \mathbb{R}^{2n_f\times 2n_l}$, and $\mathcal{L}_{ff} = (R_{\mathcal{A}}^f)^\top R_{\mathcal{A}}^f \in \mathbb{R}^{2n_f\times 2n_f}$ is defined as the *angle localization matrix* of the network $\mathbb{A}$.

Lemma 1. *For a static multi-agent network $\mathbb{A}$, one has (i) $\mathcal{L}_{ff}p_f + \mathcal{L}_{fl}p_l \equiv 0$; (ii) The network is localizable ($\hat{p}_f$ can be uniquely determined) iff $\mathcal{L}_{ff}$ is nonsingular; (iii) If the network is localizable, then p_f can be uniquely calculated by $p_f = -\mathcal{L}_{ff}^{-1}\mathcal{L}_{fl}p_l$; (iv) $\mathcal{L}_{ff}$ is nonsingular if $Rank(R_{\mathcal{A}}(\alpha(p))) = 2n - 4$.*

The proof of Lemma 1 is straightforward by following the results in Chen (2022), Chen et al. (2021), Diao et al. (2014), Fang et al. (2021), Lin et al. (2016) and Zhao and Zelazo (2016). After knowing when the network is localizable, we now introduce how to design a distributed position estimator $\dot{\hat{p}}_f(t)$ for the followers such that $\hat{p}_f(t) \to p_f$ as $t \to \infty$. Define the cost function

$$\tilde{J}(\hat{p}_f) = \hat{p}^\top\mathcal{L}(\alpha)\hat{p} = p_l^\top\mathcal{L}_{ll}(\alpha)p_l + 2p_l^\top\mathcal{L}_{lf}(\alpha)\hat{p}_f + \hat{p}_f^\top\mathcal{L}_{ff}(\alpha)\hat{p}_f.$$

Following Chen (2022), Fang et al. (2021), Zhao and Zelazo (2016), a gradient-descent localization law for the followers can be designed as

$$\dot{\hat{p}}_f(t) = -\nabla_{\hat{p}_f}\tilde{J}(\hat{p}_f) = -\mathcal{L}_{ff}(\alpha)\hat{p}_f(t) - \mathcal{L}_{fl}(\alpha)p_l \tag{7}$$

whose component form for each follower $i \in \mathcal{V}_f$ is

$$\begin{aligned} \dot{\hat{p}}_i(t) = &-\sum_{(i,j_1,k_1)\in\bar{\mathcal{A}}} (A_i^{\triangle ij_1k_1}(\alpha))^\top f_i^{\triangle ij_1k_1}(\alpha, \hat{p}(t)) \\ &-\sum_{(j_2,i,k_2)\in\bar{\mathcal{A}}} (A_i^{\triangle j_2ik_2}(\alpha))^\top f_i^{\triangle j_2ik_2}(\alpha, \hat{p}(t)) \\ &-\sum_{(j_3,k_3,i)\in\bar{\mathcal{A}}} (A_i^{\triangle j_3k_3i}(\alpha))^\top f_i^{\triangle j_3k_3i}(\alpha, \hat{p}(t)), \end{aligned} \tag{8}$$

where

$$\begin{aligned} f_i^{\triangle ij_1k_1}(\alpha, \hat{p}(t)) &= A_i^{\triangle ij_1k_1}(\alpha)\hat{p}_i(t) + A_{j_1}^{\triangle ij_1k_1}(\alpha)\hat{p}_{j_1}(t) \\ &\quad + A_{k_1}^{\triangle ij_1k_1}(\alpha)\hat{p}_{k_1}(t), \\ f_i^{\triangle j_2ik_2}(\alpha, \hat{p}(t)) &= A_{j_2}^{\triangle j_2ik_2}(\alpha)\hat{p}_{j_2}(t) + A_i^{\triangle j_2ik_2}(\alpha)\hat{p}_i(t) \\ &\quad + A_{k_2}^{\triangle j_2ik_2}(\alpha)\hat{p}_{k_2}(t), \\ f_i^{\triangle j_3k_3i}(\alpha, \hat{p}(t)) &= A_{j_3}^{\triangle j_3k_3i}(\alpha)\hat{p}_{j_3}(t) + A_{k_3}^{\triangle j_3k_3i}(\alpha)\hat{p}_{k_3}(t) \\ &\quad + A_i^{\triangle j_3k_3i}(\alpha)\hat{p}_i(t), \end{aligned}$$

and $\{j_1, j_2, j_3, k_1, k_2, k_3\} \in \mathcal{N}_i$, $\hat{p}_j(t) = p_j, \forall j \in \mathcal{V}_l$, $\bar{\mathcal{A}} \subset \mathcal{A}$, $|\bar{\mathcal{A}}| = m$ such that if $(i, j, k) \in \bar{\mathcal{A}}$, then $(j, i, k) \notin \bar{\mathcal{A}}$ and $(i, k, j) \notin \bar{\mathcal{A}}$. Since all agents are static in this localization problem, the constant matrix $A_i^{\triangle jik}(\alpha) \in \mathbb{R}^{2\times 2}$ is only related to the constant interior angles in $\triangle jik$. Moreover, (8) is distributed and can be implemented by using agent i's one-time angle measurement to obtain α_{jik}, one-time communication with agent j to obtain α_{kji} and continuous communication with agents j, k to obtain $\hat{p}_j(t)$, $\hat{p}_k(t)$, where $j \in \{j_1, j_2, j_3\}$, $k \in \{k_1, k_2, k_3\}$. Since $\mathcal{L}_{ff}(\alpha)$, $\mathcal{L}_{fl}(\alpha)$ are constant matrices, it follows straightforwardly from (7) and Lemma 1 that $\hat{p}_f(t) \to p_f$ globally as $t \to \infty$ if the triangular angularity $\mathbb{A}$ is localizable.

Different from Chen (2022), Fang et al. (2021) and Zhao and Zelazo (2016) and the localization problem introduced earlier where all static agents form a static angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, the SLAF problem to be investigated in this paper deals with dynamic angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p(t))$ since the positions $p(t)$ of the agents are time-varying. Therefore, one of the main challenges in the SLAF problem is that the interior angles formed by the agents are time-varying. According to (4) and (6), the matrices $\mathcal{L}_{ff}(\alpha(t))$ and $\mathcal{L}_{fl}(\alpha(t))$ are also time-varying, which makes the control design and stability analysis challenging due to the existence of high nonlinearity. The following three sections introduce our solutions to SLAF problems with static leaders, leaders with constant velocities, and leaders with time-varying velocities, respectively.

3. SLAF for the case of static leaders

In this section, we first formulate the problem mathematically, then design estimation laws to let the followers know their desired positions, and finally design estimation laws to localize them and design control laws to drive them from their initial positions to their desired positions, respectively.

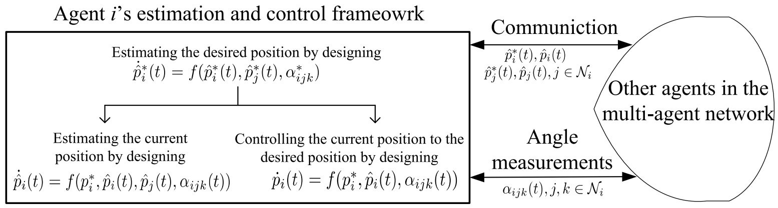


Fig. 1. The overall localization and control framework.

3.1. Problem formulation

Consider a time-varying angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p(t))$ where the leaders are static, i.e., $\dot{p}_l(t) = 0$ and the followers are dynamic, i.e., $\dot{p}_f(t) \neq 0$. The desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is localizable where p^* denotes the desired positions of all the agents which are assumed to be generic (Chen et al., 2021, Def 4), and α^* are the desired angles defined by $\mathcal{A}$. Denote by $p_l^* = p_l(0)$ the leaders' desired positions which are known by themselves, and p_f^* the followers' desired positions which are *uniquely* determined by p_l^* and α^* according to Lemma 1(iii). Consider that each follower is governed by the single-integrator dynamics

$$\dot{p}_i(t) = u_i(t), i \in \mathcal{V}_f. \tag{9}$$

The localization task is to design estimator $\dot{\hat{p}}_i(t)$ such that

$$\lim_{t\to\infty} (\hat{p}_i(t) - p_i(t)) = 0, i \in \mathcal{V}_f. \tag{10}$$

The formation task is to design control input $u_i(t)$ such that

$$\lim_{t\to\infty} (p_i(t) - p_i^*) = 0, i \in \mathcal{V}_f. \tag{11}$$

The aim of SLAF is to simultaneously achieve (10) and (11) for the followers. We consider the scenario where $\dot{\hat{p}}_i, u_i, \forall i \in \mathcal{V}_f$ are only the functions of $\alpha_{ijk}(t), \alpha_{jik}(t), \alpha^*_{ijk}, \alpha^*_{jik}, \hat{p}_i(t), \hat{p}_j(t), \hat{p}_i^*(t), \hat{p}_j^*(t)$, $\forall j, k \in \mathcal{N}_i$, i.e., the real-time angle measurements and the desired angles formed with the neighbors, the real-time estimates of own and the neighbors' positions, and the real-time estimates of own and the neighbors' desired positions. The main strategy of our solution is to simultaneously estimate each follower's real-time position and drive the follower from its current position to its desired position. Since each follower i does not know its desired position p_i^* but knows the desired angles α^*_{ijk} with respect to its neighbors j, k, each follower i needs to estimate its desired position p_i^* by using the information of $\alpha^*_{ijk}, \alpha^*_{jik}$ firstly.

3.2. Estimation of the followers' desired positions within finite time

In this subsection, we design estimation law $\dot{\hat{p}}_i^*(t)$ using the information of $\alpha^*_{ijk}, \hat{p}_i^*(t), \hat{p}_j^*(t), \forall (i, j, k) \in \mathcal{A}$ such that each follower i can obtain p_i^* within finite time. Towards this end, we design the estimation law as

$$\dot{\hat{p}}_f^*(t) = -\mathrm{sig}^{\beta_1}\left(\mathcal{L}_{ff}(\alpha^*)\hat{p}_f^*(t) + \mathcal{L}_{fl}(\alpha^*)p_l\right) \tag{12}$$

whose component form for each follower $i \in \mathcal{V}_f$ is

$$\begin{aligned}\dot{\hat{p}}_i^*(t) = -\mathrm{sig}^{\beta_1}[&\sum_{(i,j_1,k_1)\in\bar{\mathcal{A}}} (A_i^{\triangle ij_1k_1}(\alpha^*))^\top f_i^{\triangle ij_1k_1}(\alpha^*, \hat{p}^*(t)) \\ + &\sum_{(j_2,i,k_2)\in\bar{\mathcal{A}}} (A_i^{\triangle j_2ik_2}(\alpha^*))^\top f_i^{\triangle j_2ik_2}(\alpha^*, \hat{p}^*(t)) \\ + &\sum_{(j_3,k_3,i)\in\bar{\mathcal{A}}} (A_i^{\triangle j_3k_3i}(\alpha^*))^\top f_i^{\triangle j_3k_3i}(\alpha^*, \hat{p}^*(t))]\end{aligned} \tag{13}$$

where $\hat{p}_j^*(t) = p_j^*, \forall j \in \mathcal{V}_l$, $0 < \beta_1 < 1$, $f_i^{\triangle ij_1k_1}(\alpha^*, \hat{p}^*(t)) = A_i^{\triangle ij_1k_1}(\alpha^*)\hat{p}_i^*(t) + A_{j_1}^{\triangle ij_1k_1}(\alpha^*)\hat{p}_{j_1}^*(t) + A_{k_1}^{\triangle ij_1k_1}(\alpha^*)\hat{p}_{k_1}^*(t)$, and $\hat{p}_i^*(t)$ denotes agent i's estimate of p_i^*. Since the designed estimation law (13) only uses the local information of $\alpha^*_{ijk}, \hat{p}_i^*(t)$ and the communication information of $\hat{p}_j^*(t), j \in \mathcal{N}_i$ from its neighbor j, it is distributed and is independent of agents' real-time positions $p(t)$. Then, one has the following results.

Theorem 1. *If the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is localizable, then $\hat{p}_f^*(t)$ under (12) converges to p_f^* within finite time.*

Proof. By using Lemma 1, $\mathcal{L}_{ff}(\alpha^*)$ is nonsingular and positive definite. Design Lyapunov function candidate

$$V_1(t) = 0.5\tilde{p}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)\tilde{p}_f^*(t), \tag{14}$$

where $\tilde{p}_f^*(t) = \hat{p}_f^*(t) - p_f^*$. Taking the time-derivative of (14) yields

$$\begin{aligned}\dot{V}_1(t) =& \tilde{p}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)\dot{\hat{p}}_f^*(t) \\ =& -\tilde{p}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)\mathrm{sig}^{\beta_1}\left(\mathcal{L}_{ff}(\alpha^*)\hat{p}_f^*(t) + \mathcal{L}_{fl}(\alpha^*)p_l\right) \\ =& -\tilde{p}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)\mathrm{sig}^{\beta_1}\left(\mathcal{L}_{ff}(\alpha^*)\tilde{p}_f^*(t)\right),\end{aligned} \tag{15}$$

where we have used the fact $p_f^* = -\mathcal{L}_{ff}^{-1}(\alpha^*)\mathcal{L}_{fl}(\alpha^*)p_l$ by following Lemma 1. It follows that

$$\begin{aligned}\dot{V}_1(t) &= -(\mathcal{L}_{ff}(\alpha^*)\tilde{p}_f^*(t))^\top \mathrm{sig}^{\beta_1}\left(\mathcal{L}_{ff}(\alpha^*)\tilde{p}_f^*(t)\right) \\ &\le -c_1 V_1^{\frac{\beta_1+1}{2}}(t),\end{aligned}$$

where $c_1 = 2^{\frac{\beta_1+1}{2}}(\lambda_{\min}^{\beta_1+1}(\mathcal{L}_{ff}(\alpha^*))/\lambda_{\max}^{\frac{\beta_1+1}{2}}(\mathcal{L}_{ff}(\alpha^*))) > 0$, $0 < \frac{\beta_1+1}{2} < 1$. According to Hong, Huang, and Xu (2001, Lemma1), one has that there exists a settling time $T_1 = \frac{2}{c_1(1-\beta_1)}V_1^{\frac{1-\beta_1}{2}}(0)$ such that for $\forall t > T_1$, $V_1(t) = 0$, i.e., $\hat{p}_f^*(t) = p_f^*$. □

After the followers have the knowledge of their desired positions by (12), we then design position estimation laws and formation control laws to simultaneously achieve (10) and (11). The overall localization and control framework is shown in Fig. 1.

Remark 2. In this SLAF problem, the leaders are required to know their absolute positions in the global coordinate frame, which can be fulfilled by equipping GPS receivers. In the case where GPS signal is unavailable, the leaders can establish a temporary global coordinate frame using local relative position measurements with respect to some common environmental landmarks, by which the leaders can obtain their coordinates with respect to the temporary global coordinate frame. In addition, according to (4) and Lemma 1, the minimum number of angle constraints in $\mathcal{A}$ to make $\mathbb{A}^*$ localizable is $2n - 4$, which is also the minimum number of required angle measurements to conduct the SLAF task.

3.3. Simultaneous localization and formation

Without requiring the knowledge of agents' absolute positions or relative positions with respect to their neighbors, we design the simultaneous localization and formation algorithm for the followers as

$$\dot{\hat{p}}_f(t) = -\mathcal{L}_{ff}(\alpha(t))\hat{p}_f(t) - \mathcal{L}_{fl}(\alpha(t))p_l + u_f(t) - k_1(\hat{p}_f(t) - \hat{p}_f^*(t)), \quad (16)$$

$$\dot{p}_f(t) = u_f(t) = -k_1(\hat{p}_f(t) - \hat{p}_f^*(t)), \quad (17)$$

where k_1 is a positive scalar, $\alpha(p(t))$ is abbreviated as $\alpha(t)$, and $\hat{p}_f^*$ is governed by (12). Since $p_f(t)$ is time-varying under (16) and (17), the angles $\alpha(t)$ among the leaders and the followers and the matrices $\mathcal{L}_{ff}(\alpha(t))$, $\mathcal{L}_{fl}(\alpha(t))$ are all time-varying. The component form of the simultaneous localization and formation algorithm (16) and (17) can be written as

$$\begin{aligned}\dot{\hat{p}}_i(t) = &- \sum_{(i,j_1,k_1)\in\bar{\mathcal{A}}} (A_i^{\triangle ij_1k_1}(\alpha(t)))^\top f_i^{\triangle ij_1k_1}(\alpha(t),\hat{p}(t)) \\ &- \sum_{(j_2,i,k_2)\in\bar{\mathcal{A}}} (A_i^{\triangle j_2ik_2}(\alpha(t)))^\top f_i^{\triangle j_2ik_2}(\alpha(t),\hat{p}(t)) \\ &- \sum_{(j_3,k_3,i)\in\bar{\mathcal{A}}} (A_i^{\triangle j_3k_3i}(\alpha(t)))^\top f_i^{\triangle j_3k_3i}(\alpha(t),\hat{p}(t)) \\ &+ u_i(t) - k_1(\hat{p}_i(t) - \hat{p}_i^*(t)), \end{aligned} \quad (18)$$

$$\dot{p}_i(t) = u_i(t) = -k_1(\hat{p}_i(t) - \hat{p}_i^*(t)), \quad i \in \mathcal{V}_f, \quad (19)$$

where $f_i^{\triangle ij_1k_1}(\alpha(t),\hat{p}(t)) = A_i^{\triangle ij_1k_1}(\alpha(t))\hat{p}_i(t) + A_{j_1}^{\triangle ij_1k_1}(\alpha(t))\hat{p}_{j_1}(t) + A_{k_1}^{\triangle ij_1k_1}(\alpha(t))\hat{p}_{k_1}(t)$ and $\hat{p}_j(t) = p_j^*, \forall j \in \mathcal{V}_l$. According to the component form of the SLAF algorithm for agent i, one can see that (18) and (19) only need the angle measurement information $\alpha_{jik}(t)$, $\alpha_{ijk}(t)$ and the communication information $\hat{p}_j(t)$, $\forall j \in \mathcal{N}_i$. Therefore, the simultaneous localization and formation algorithm (16) and (17) is distributed and can be implemented using local angle measurements with respect to neighbors and communication information from neighbors. Before giving stability analysis, we need an assumption such that (16) is well-defined.

Assumption 1. No collision occurs among neighboring agents, i.e., $p_i(t) \neq p_j(t)$, $p_j(t) \neq p_k(t)$ and $p_i(t) \neq p_k(t)$ for $\forall (j,i,k) \in \mathcal{A}$ and $t \geq 0$.

Due to the absence of inter-agent distance measurements, Assumption 1 is a common assumption required also in other bearing-only or angle-only formation control problems (Chen et al., 2021; Zhao, Li, & Ding, 2019; Zhao & Zelazo, 2016). Given a specific formation control law, inter-agent collision is determined by the initial formation (Zhao et al., 2019). In this paper, we will also present sufficient conditions on the agents' initial states to guarantee both collision avoidance and system convergence.

Now, we conduct the stability analysis for (16) and (17). First, we aim to obtain that $\dot{\hat{p}}_f(t)$, $u_f(t)$ are bounded for $t \leq T_1$. According to the proof of Theorem 1, $\hat{p}_f^*(t)$ is always bounded for $\forall t \leq T_1$ since its evolution is independent of (16)–(17). Then, under Assumption 1 and bounded initial states $\hat{p}_f(0)$, $p_f(0)$, one has that $\dot{\hat{p}}_f(t)$, $u_f(t)$ in (16)–(17) are bounded for $t \leq T_1$. This is because $\mathcal{L}_{ff}(\alpha(t))$, $\mathcal{L}_{fl}(\alpha(t))$ are functions of $\sin\alpha_{jik}(t)$ and $\cos\alpha_{jik}(t)$, $(j,i,k) \in \mathcal{A}$, which are bounded for any angles $\alpha_{jik}(t)$ satisfying Assumption 1. Therefore, $p_f(t)$, $\hat{p}_f(t)$ will not diverge for $t \leq T_1$. In the following, we analyze the stability of (16)–(17) for $t > T_1$, in which $\hat{p}_f^*(t) = p_f^*$. Define $\tilde{p}_f(t) := (p_f(t) - p_f^*) \in \mathbb{R}^{2n_f}$ as the position-based formation error, and $\tilde{p}_{ef}(t) := (\hat{p}_f(t) - p_f(t)) \in \mathbb{R}^{2n_f}$ as the position estimation error. The dynamics of $\tilde{p}_f(t)$ and $\tilde{p}_{ef}(t)$ under the algorithms (16) and (17) can be written as

$$\begin{bmatrix}\dot{\tilde{p}}_{ef}(t) \\ \dot{\tilde{p}}_f(t)\end{bmatrix} = -\begin{bmatrix}\mathcal{L}_{ff}(\alpha(t)) + k_1 I_{2n_f} & k_1 I_{2n_f} \\ k_1 I_{2n_f} & k_1 I_{2n_f}\end{bmatrix}\begin{bmatrix}\tilde{p}_{ef}(t) \\ \tilde{p}_f(t)\end{bmatrix}, \quad (20)$$

where we have used the facts that $\mathcal{L}_{ff}(\alpha(t))p_f(t) + \mathcal{L}_{fl}(\alpha(t))p_l = 0$ and $\hat{p}_f(t) - p_f^* = \tilde{p}_f(t) + \tilde{p}_{ef}(t)$. Compared to the localization dynamics (7) for static angularities, the simultaneous localization and formation dynamics (20) for dynamic angularities are more complicated because the system matrix in (20) is state-dependent and highly nonlinear. Now, we present the results on the closed-loop dynamics (20).

Theorem 2. *Consider a multi-agent system consisting of static leaders and dynamic followers governed by (9). Suppose that each follower $i \in \mathcal{V}_f$ has angle measurements $\alpha_{jik}(t)$ and the information $\hat{p}_j^*(t)$, $\hat{p}_j(t)$, $\alpha_{ijk}(t)$ obtained from the communication with its neighbor $j \in \mathcal{N}_i$. Under the SLAF algorithms (16)–(17), if Assumption 1 holds and the desired triangular angularity $\mathbb{A}^*(\mathcal{V},\mathcal{A},p^*)$ is localizable, then the following conclusions hold:*

(i) The equilibrium set of the dynamics (20) is

$$\begin{aligned}\Omega_{es} &= \{(\tilde{p}_{ef},\tilde{p}_f) |\ R_{\mathcal{A}}^f(\alpha)\tilde{p}_{ef} = 0,\ \tilde{p}_{ef} + \tilde{p}_f = 0\} \\ &= \{(\tilde{p}_{ef},\tilde{p}_f) |\ R_{\mathcal{A}}^f(\alpha)(p_f - p_f^*) = 0,\ \hat{p}_f = p_f^*\}\end{aligned} \quad (21)$$

Moreover, no matter what the initial states $\tilde{p}_{ef}(0)$, $\tilde{p}_f(0)$ are, the closed-loop dynamics (20) always converge to the set Ω_{es}.

(ii) The desired equilibrium $\Omega_{es1} = \{(\tilde{p}_{ef},\tilde{p}_f) |\ \tilde{p}_{ef} = 0, \tilde{p}_f = 0\}$ of (20) is locally and exponentially stable.

(iii) The desired equilibrium of the closed-loop dynamics (20) is exponentially stable iff there exist some positive scalars T_2, γ_1, γ_2 such that for $\forall t > 0$

$$\gamma_1 I_{2n_f} \leq \int_t^{t+T_2} \begin{bmatrix}\mathcal{L}_{ff}(\alpha(\tau)) + k_1 I_{2n_f} & k_1 I_{2n_f} \\ k_1 I_{2n_f} & k_1 I_{2n_f}\end{bmatrix} d\tau \leq \gamma_2 I_{2n_f}. \quad (22)$$

Moreover, if no collinearity occurs $\forall t > 0$ for all the triangles defined in $\mathcal{A}$, then the desired equilibrium of (20) is asymptotically stable.

Proof of Theorem 2. (i): Letting $\dot{\tilde{p}}_{ef} = 0$, $\dot{\tilde{p}}_f = 0$ in (20), one has that

$$\begin{cases}\mathcal{L}_{ff}(\alpha)\tilde{p}_{ef} + k_1\tilde{p}_{ef} + k_1\tilde{p}_f = 0 \\ k_1\tilde{p}_{ef} + k_1\tilde{p}_f = 0\end{cases} \quad (23)$$

which implies that $\tilde{p}_{ef} + \tilde{p}_f = \hat{p}_f - p_f^* = 0$ and $\mathcal{L}_{ff}(\alpha)\tilde{p}_{ef} = 0$. According to (6), one has $\tilde{p}_{ef}^\top\mathcal{L}_{ff}(\alpha)\tilde{p}_{ef} = (R_{\mathcal{A}}^f(\alpha)\tilde{p}_{ef})^\top R_{\mathcal{A}}^f(\alpha)\tilde{p}_{ef} = 0$ which implies that $R_{\mathcal{A}}^f(\alpha)\tilde{p}_{ef} = 0$. To prove that the solutions of (20) starting from arbitrary initial conditions will converge to Ω_{es}, we design the Lyapunov function candidate as

$$V_2(t) = 0.5\tilde{p}_{ef}^\top(t)\tilde{p}_{ef}(t) + 0.5\tilde{p}_f^\top(t)\tilde{p}_f(t) \quad (24)$$

which is positive definite and radially unbounded. Taking the time-derivative of $V_2(t)$ along the closed-loop dynamics (20) yields

$$\begin{aligned}\dot{V}_2(t) &= \tilde{p}_{ef}^\top(t)\dot{\tilde{p}}_{ef}(t) + \tilde{p}_f^\top(t)\dot{\tilde{p}}_f(t) \\ &= -\tilde{p}_{ef}^\top(t)\mathcal{L}_{ff}(\alpha(t))\tilde{p}_{ef}(t) - k_1\tilde{p}_{ef}^\top(t)\tilde{p}_{ef}(t) \\ &\quad - 2k_1\tilde{p}_{ef}^\top(t)\tilde{p}_f(t) - k_1\tilde{p}_f^\top(t)\tilde{p}_f(t) \\ &= -\tilde{p}_{ef}^\top(t)\mathcal{L}_{ff}(\alpha(t))\tilde{p}_{ef}(t) - k_1\|\tilde{p}_{ef}(t) + \tilde{p}_f(t)\|^2 \leq 0\end{aligned} \quad (25)$$

where we have used the fact that $\mathcal{L}_{ff}(\alpha(t))$ is positive semi-definite for any $p(t) \in \mathbb{R}^{2n}$ satisfying Assumption 1. Since $V_2(t) > 0$, $\dot{V}_2(t) \leq 0$, according to LaSalle's invariance principle (Khalil, 2002, Corollary 4.2), one has that all the solutions of (20) converge to the set $\{(\tilde{p}_{ef},\tilde{p}_f) |\ \dot{V}_2(t) = 0\} = \{(\tilde{p}_{ef},\tilde{p}_f) |\ R_{\mathcal{A}}^f(\alpha)\tilde{p}_{ef} = 0,\ \tilde{p}_{ef} + \tilde{p}_f = 0\}$. □

Proof of Theorem 2. (ii): To conduct linearization for local stability analysis, we consider $\mathbb{U} = \{Y | \|Y\| \leq r_e\}$ as the small neighborhood of the desired equilibrium $Y = [\tilde{p}_{ef}^\top, \tilde{p}_f^\top]^\top = 0$, where r_e is chosen to be small such that inside $\mathbb{U}$, there are no configurations with inter-agent collinearity or collision. Note that such kind of r_e always exists because $Y = 0$ corresponds to the desired formation configuration (which is generic, i.e., without inter-agent collinearity and collision) and an arbitrary configuration in $\mathbb{U}$ is close to the desired configuration. Since $\dot{V}_2(t) \le 0$ for all $Y \in \mathbb{U}$, $\mathbb{U}$ is a positively invariant set under (20). When $Y(0) \in \mathbb{U}$, linearizing $\tilde{p}_f$ in (20) around the desired equilibrium Ω_{es1} yields

$$\begin{aligned}\dot{\tilde{p}}_f(t) =& \left[\frac{\partial(-k_1\tilde{p}_{ef}(t) - k_1\tilde{p}_f(t))}{\partial \tilde{p}_{ef}(t)}|_{\tilde{p}_{ef}(t)=0,\tilde{p}_f=0}\right]\tilde{p}_{ef}(t) \\ &+ \left[\frac{\partial(-k_1\tilde{p}_{ef}(t) - k_1\tilde{p}_f(t))}{\partial \tilde{p}_f(t)}|_{\tilde{p}_{ef}(t)=0,\tilde{p}_f=0}\right]\tilde{p}_f(t) \\ =& -k_1\tilde{p}_{ef}(t) - k_1\tilde{p}_f(t). \end{aligned} \tag{26}$$

Since $\frac{\partial \cos\alpha_{jik}}{\partial p_i} = \frac{\partial(b_{ij}^\top b_{ik})}{\partial p_i} = \frac{b_{ik}^\top(I_2 - b_{ij}b_{ij}^\top)}{\|p_i-p_j\|} + \frac{b_{ij}^\top(I_2 - b_{ik}b_{ik}^\top)}{\|p_i-p_k\|}$ and $\frac{\partial \sin\alpha_{jik}}{\partial p_i} = \frac{\partial(b_{ik}^\top b_{ij}^\perp)}{\partial p_i} = \frac{b_{ij}^{\perp\top}(I_2 - b_{ik}b_{ik}^\top)}{\|p_i-p_k\|} + \frac{b_{ik}^\top(I_2 - b_{ij}^\perp b_{ij}^{\perp\top})}{\|p_i-p_j\|}$, the partial derivatives of the elements in $\mathcal{L}_{ff}(\alpha(t))$ with respect to p_f are well-defined and continuous for all $Y \in \mathbb{U}$. Following Khalil (2002, Section4.3), the linearized dynamics of $\dot{\tilde{p}}_{ef}$ is

$$\dot{\tilde{p}}_{ef}(t) = -\mathcal{L}_{ff}(\alpha^*)\tilde{p}_{ef}(t) - k_1\tilde{p}_{ef}(t) - k_1\tilde{p}_f(t), \tag{27}$$

where we have also used the fact $\mathcal{L}_{ff}(\alpha(t))|_{\tilde{p}_{ef}(t)=0,\tilde{p}_f=0} = \mathcal{L}_{ff}(\alpha^*)$. Writing (26)–(27) into a compact form yields

$$\begin{bmatrix}\dot{\tilde{p}}_{ef}(t) \\ \dot{\tilde{p}}_f(t)\end{bmatrix} = -\begin{bmatrix}\mathcal{L}_{ff}(\alpha^*) + k_1 I_{2n_f} & k_1 I_{2n_f} \\ k_1 I_{2n_f} & k_1 I_{2n_f}\end{bmatrix}\begin{bmatrix}\tilde{p}_{ef}(t) \\ \tilde{p}_f(t)\end{bmatrix}. \tag{28}$$

The characteristic polynomial of (28) is

$$\begin{aligned}&\det[(\lambda I_{2n_f} + \mathcal{L}_{ff}(\alpha^*) + k_1 I_{2n_f})(\lambda + k_1) - k_1^2 I_{2n_f}] \\ &= \prod_{i=1}^{2n_f}\left(\lambda^2 + (2k_1 + \lambda_i(\mathcal{L}_{ff}(\alpha^*)))\lambda + k_1\lambda_i(\mathcal{L}_{ff}(\alpha^*))\right) \\ &= 0 \end{aligned} \tag{29}$$

where $\lambda_i(\mathcal{L}_{ff}(\alpha^*))$ denotes the ith eigenvalue of $\mathcal{L}_{ff}(\alpha^*)$, $1 \le i \le 2n_f$ in a descending order. Since $\mathcal{L}_{ff}(\alpha^*)$ is positive definite, all the solutions of (29) are negative. Following the Lyapunov analysis in the proof of Khalil (2002, Theorem 4.7), one has that the desired equilibrium of (20) is locally and exponentially stable. □

To prove Theorem 2. (iii), we first introduce a lemma about the persistent excitation condition which is similarly used in Ye et al. (2017) and Han et al. (2018).

Lemma 2 (*Anderson (1977)*). *Let $W(\cdot) : \mathbb{R}^+ \to \mathbb{R}^{m\times r}$ be regulated. Then, the equilibrium of*

$$\dot{x}(t) = -W(t)W^\top(t)x(t) \tag{30}$$

is globally and exponentially stable iff there exist positive scalars T_2, γ_1, γ_2 such that for $\forall t > 0$, $\gamma_1 I_{m\times m} \le \int_t^{t+T_2}\left(W(\tau)W^\top(\tau)\right)d\tau \le \gamma_2 I_{m\times m}$. □

Proof of Theorem 2. (iii): Note that $\begin{bmatrix}\mathcal{L}_{ff}(\alpha(t)) + k_1 I_{2n_f} & k_1 I_{2n_f} \\ k_1 I_{2n_f} & k_1 I_{2n_f}\end{bmatrix} = \begin{bmatrix}(R_{\mathcal{A}}^f(\alpha(t)))^\top & \sqrt{k_1}I_{2n_f} \\ 0 & \sqrt{k_1}I_{2n_f}\end{bmatrix}\begin{bmatrix}(R_{\mathcal{A}}^f(\alpha(t)))^\top & \sqrt{k_1}I_{2n_f} \\ 0 & \sqrt{k_1}I_{2n_f}\end{bmatrix}^\top$. Using Lemma 2, the desired equilibrium of (20) is globally and exponentially stable iff (22) holds. Moreover, if no collinearity occurs $\forall t > 0$ for all the triangles defined in $\mathcal{A}$, then $\mathcal{L}_{ff}(\alpha(t))$ is always positive definite for $\forall t > 0$ since $\mathbb{A}^*$ is localizable. Therefore, $\tilde{p}_{ef}^\top(t)\mathcal{L}_{ff}(\alpha(t))\tilde{p}_{ef}(t) > 0$ and $\dot{V}_2(t) < 0$ by using (25). It follows from Khalil (2002, Theorem 4.1) that the desired equilibrium of (20) is asymptotically stable. □

The overall equilibrium set Ω_{es} in (21) consists of not only the desired equilibrium set Ω_{es1} where the followers reach their target positions, but also the undesired equilibrium set $\Omega_{es2} = \Omega_{es} - \Omega_{es1} = \{(\tilde{p}_{ef}, \tilde{p}_f) | R_{\mathcal{A}}^f(\alpha)(p_f - p_f^*) = 0, p_f \ne p_f^*, \ \hat{p}_f = p_f^*\}$ where the followers become static but they have not reached their target positions. Clearly, Ω_{es2} is much larger than Ω_{es1}, which makes its analysis vital. Now, consider in the system evolution of (20) that $[\tilde{p}_{ef}^\top, \tilde{p}_f^\top]^\top$ falls into Ω_{es2}. According to the facts $R_{\mathcal{A}}^f(\alpha)(p_f - p_f^*) = 0$ and $p_f \ne p_f^*$ in the definition of Ω_{es2}, one has $\mathrm{Rank}(R_{\mathcal{A}}^f(\alpha)) < 2n_f$. Since $\mathrm{Rank}(\mathcal{L}_{ff}(\alpha)) = \mathrm{Rank}(R_{\mathcal{A}}^f(\alpha))$, $\mathcal{L}_{ff}(\alpha(p))$ is singular and the whole angularity becomes unlocalizable when $[\tilde{p}_{ef}^\top, \tilde{p}_f^\top]^\top \in \Omega_{es2}$. Since the leaders are static and $\mathbb{A}^*$ is localizable, at least one triangle defined in $\mathcal{A}$ is degenerate Chen (2022), i.e., collinear, when $[\tilde{p}_{ef}^\top, \tilde{p}_f^\top]^\top$ reaches one of those undesired equilibrium in Ω_{es2}. Next, we use a perturbation-based approach to drive the system away from Ω_{es2} when the system reaches an undesired equilibrium, i.e., $[\tilde{p}_{ef}^\top(t), \tilde{p}_f^\top(t)]^\top \in \Omega_{es2}$, at some time instants t.

3.4. Perturbation-based simultaneous localization and formation with asymptotic convergence

Inspired by Trinh, Zelazo, and Ahn (2019), we design a perturbation-based algorithm such that the system can escape away from collinear or unlocalizable configurations when they reach Ω_{es2}. Towards this end, we modify the SLAF algorithm (18)–(19) into

$$\begin{aligned}\dot{\hat{p}}_i(t) =& -\sum_{(i,j_1,k_1)\in\bar{\mathcal{A}}}(A_i^{\triangle ij_1k_1}(\alpha(t)))^\top f_i^{\triangle ij_1k_1}(\alpha(t), \hat{p}(t)) \\ &-\sum_{(j_2,i,k_2)\in\bar{\mathcal{A}}}(A_i^{\triangle j_2ik_2}(\alpha(t)))^\top f_i^{\triangle j_2ik_2}(\alpha(t), \hat{p}(t)) \\ &-\sum_{(j_3,k_3,i)\in\bar{\mathcal{A}}}(A_i^{\triangle j_3k_3i}(\alpha(t)))^\top f_i^{\triangle j_3k_3i}(\alpha(t), \hat{p}(t)) \\ &+ u_i(t) - k_1(\hat{p}_i(t) - p_i^*) - e_i(t)\left(\mathrm{sgn}(\hat{p}_i(t) - p_i^*) - \delta_i(t)\right), \end{aligned} \tag{31}$$

$$\begin{aligned}\dot{p}_i(t) = u_i(t) =& -k_1(\hat{p}_i(t) - p_i^*) \\ &- e_i(t)\left(\mathrm{sgn}(\hat{p}_i(t) - p_i^*) - \delta_i(t)\right), \ i \in \mathcal{V}_f \end{aligned} \tag{32}$$

where $e_i = \sum_{(j_4,i,k_4)\in\mathcal{A}}|\alpha_{j_4ik_4}(t) - \alpha_{j_4ik_4}^*| \ge 0$ is the sum of the absolute values of the angle errors associated with agent i, and the perturbation term $\delta_i(t) = [\delta_{i1}(t), \delta_{i2}(t)]^\top \in \mathbb{R}^2$ is a continuous time-varying vector satisfying $\|\delta_i(t)\| < 1$, and we considered $t > T_1$.

Theorem 3. *Consider a multi-agent system consisting of static leaders and followers governed by (9). Suppose that each follower $i \in \mathcal{V}_f$ has angle measurements $\alpha_{jik}(t)$ and the information $\hat{p}_j^*(t), \hat{p}_j(t), \alpha_{ijk}(t)$ obtained from the communication with its neighbor $j \in \mathcal{N}_i$. Under the simultaneous localization and formation algorithm (31) and (32), if Assumption 1 holds and the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is localizable, then the multi-agent system asymptotically converges to the desired equilibrium Ω_{es1}.*

Proof. Since the right sides of (31)–(32) are discontinuous, we consider the solutions $\hat{p}_f(t), p_f(t)$ of (31)–(32) in the Filippov sense. Because the signum function is measurable and locally essentially bounded, the Filippov solutions of (31)–(32) exist (Filipov, 1988). Consider the same Lyapunov function candidate $V_2(t)$ for (31) and (32), which is continuously differentiable and radially unbounded. Then, $\dot{V}_2(t)$ exists almost everywhere (abbreviated as a.e.) and $\dot{V}_2 \in^{a.e.} \dot{\tilde{V}}_2$ where $\in$ represents the differential inclusion here, and

$$\begin{aligned}\dot{\tilde{V}}_2(t) &= \tilde{p}_{ef}^{\top}\dot{\tilde{p}}_{ef} + \tilde{p}_f^{\top}\dot{\tilde{p}}_f \\ &= -\tilde{p}_{ef}^{\top}(t)\mathcal{L}_{ff}(\alpha(t))\tilde{p}_{ef}(t) - k_1\|\tilde{p}_{ef}(t)+\tilde{p}_f(t)\|^2 \\ &- \sum_{i=n_a+1}^{n_a+n_f} e_i(t)(\hat{p}_i(t)-p_i(t))^{\top}\left(K[\mathrm{sgn}](\hat{p}_i(t)-p_i^*)-\delta_i(t)\right) \\ &- \sum_{i=n_a+1}^{n_a+n_f} e_i(t)\left(p_i(t)-p_i^*\right)^{\top}(K[\mathrm{sgn}](\hat{p}_i(t)-p_i^*)-\delta_i(t)).\end{aligned} \tag{33}$$

and $K[f](x)$ represents the Filippov set-valued mapping of a vector function $f(x)$ (Trinh et al., 2019). Note that the last two components in (33) can be rewritten as

$$\begin{aligned}&- \sum_{i=n_a+1}^{n_a+n_f} e_i(t)(\hat{p}_i(t)-p_i^*)^{\top}(K[\mathrm{sgn}](\hat{p}_i(t)-p_i^*)-\delta_i(t)) \\ &= -\sum_{i=n_a+1}^{n_a+n_f} e_i(t)(\|\hat{p}_i(t)-p_i^*\|_1 - (\hat{p}_i(t)-p_i^*)^{\top}\delta_i(t)) \\ &\le -\sum_{i=n_a+1}^{n_a+n_f} e_i(t)\|\hat{p}_i(t)-p_i^*\|_1(1-\|\delta_i(t)\|) \le 0\end{aligned} \tag{34}$$

where we have used the facts that $e_i \ge 0$ is a scalar, $(\hat{p}_i(t)-p_i^*)^{\top}K[\mathrm{sgn}](\hat{p}_i(t)-p_i^*) = \|\hat{p}_i(t)-p_i^*\|_1$ (Filipov, 1988; Trinh et al., 2019), and $(\hat{p}_i(t)-p_i^*)^{\top}\delta_i(t) \le \|\hat{p}_i(t)-p_i^*\|_2\|\delta_i(t)\|_2 \le \|\hat{p}_i(t)-p_i^*\|_1\|\delta_i(t)\|_2$. Substituting (34) into (33) yields

$$\dot{\tilde{V}}_2(t) \le -\tilde{p}_{ef}^{\top}(t)\mathcal{L}_{ff}(\alpha(t))\tilde{p}_{ef}(t) - k_1\|\tilde{p}_{ef}(t)+\tilde{p}_f(t)\|^2$$

which implies that the solutions of (31) and (32) will converge to the set $\Omega_{es} = \Omega_{es1} \cup \Omega_{es2}$. Now, we prove that a state in Ω_{es2} is *not* an equilibrium of (31)–(32). If an arbitrary state $[\tilde{p}_{ef}^{\top}, \tilde{p}_f^{\top}]^{\top} \in \Omega_{es2}$, then the formation configuration of this state has at least one degenerate triangle. Therefore, $\exists i \in \mathcal{V}_f$ such that $e_i \ne 0$. Since $\delta_i(t)$ is a time-varying perturbation, $e_i\left(\mathrm{sgn}(\hat{p}_i - p_i^*) - \delta_i\right) \ne 0$ which implies $\dot{p}_i \ne 0, \dot{\hat{p}}_i \ne 0$, i.e., an arbitrary state in Ω_{es2} is *not* an equilibrium of (31)–(32). However, Ω_{es1} is still an equilibrium of (31)–(32). Using LaSalle's invariance principle for nonsmooth systems (Fischer, Kamalapurkar, & Dixon, 2013; Trinh et al., 2019), the state under (31)–(32) asymptotically converges to the desired equilibrium Ω_{es1}. □

Remark 3. Compared to Trinh et al. (2019) where the perturbation is pair-wise, the perturbation $\delta_i(t)$ we used in (31)–(32) can be selected individually in each agent. Although the perturbation $\delta_i(t)$ has positive effect of driving the system away from unlocalizable configurations, it may have negative effect on the system's convergence rate. To keep the perturbation sufficiently small when it is unnecessary, one can choose $\delta_i(t) = e^{-\gamma_3 t}\begin{bmatrix}\cos t \\ \sin t\end{bmatrix}$ where γ_3 is a positive scalar. In addition, according to (3), $\mathcal{L}_{ff}(\alpha(t))$ in (20) is a function of $\sin\alpha_{jik}(p(t))$ and $\cos\alpha_{jik}(p(t))$, $(j,i,k) \in \mathcal{A}$, which implies that $\mathcal{L}_{ff}(\alpha(t)), \mathcal{L}_{fl}(\alpha(t))$ are state-dependent, well-defined, and continuous even when a collinearity occurs among neighboring agents. Therefore, if Assumption 1 holds, the systems (16)–(17) and (31)–(32) have solutions.

4. SLAF for the case of leaders with constant moving velocities

In this section, we consider that the leaders are dynamic and move with constant velocities, i.e., $\dot{p}_i^*(t) = v_c^* \ne 0, t \ge 0, i \in \mathcal{V}_l$ where $v_c^* \in \mathbb{R}^2$ is a constant.

4.1. Problem formulation

Consider that the desired angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*(t))$ is time-varying where $p^*(t)$ denotes the desired positions of all the agents, and α^* are the desired constant angles defined by $\mathcal{A}$, and the desired formation shape is specified by α^*. Then, the leaders' desired positions $p_l^*(t)$ can be written as

$$p_l^*(t) = p_l(t) = p_l^*(0) + (1_{n_l} \otimes v_c^*)t, \ \forall t \ge 0 \tag{35}$$

where $v_c^* \in \mathbb{R}^2$ is the constant velocity of the leaders. Our aim still consists of simultaneously achieving localization (10) and formation (11) using the information of $\alpha_{ijk}(t), \alpha_{ijk}^*, \hat{p}_i(t), \hat{p}_j(t), \hat{p}_i^*(t), \hat{p}_j^*(t), \forall(i,j,k) \in \mathcal{A}$.

4.2. Estimation of the followers' desired positions

To obtain the followers' desired positions, we design the estimation law as

$$\dot{\hat{p}}_f^*(t) = -\beta_2 \mathrm{sgn}\left(\mathcal{L}_{ff}(\alpha^*)\hat{p}_f^*(t) + \mathcal{L}_{fl}(\alpha^*)p_l^*(t)\right), \tag{36}$$

where $\beta_2 > \sqrt{n_f}\|v_c^*\|$ and $\hat{p}_f^*(t)$ denotes the followers' estimation of $p_f^*(t)$. According to the component form (13), each follower i under the designed estimation law (36) only uses the desired information $\alpha_{ijk}^*, \alpha_{kij}^*$, the communication information $\hat{p}_j^*(t), j \in \mathcal{N}_i$ from its neighbors, the upper bound of the desired moving velocity v_c^*, and the number of the followers. Then, one has the following results.

Theorem 4. *If the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*(t))$ is localizable, then $\hat{p}_f^*(t)$ under (12) converges to $p_f^*(t)$ within finite time.*

Proof. According to Lemma 1. (i), one has that $p_f^*(t)$ can be calculated by $p_f^*(t) = -\mathcal{L}_{ff}^{-1}(\alpha^*)\mathcal{L}_{fl}(\alpha^*)p_l^*(t)$. Then, we design the Lyapunov function candidate as

$$V_3(t) = 0.5\tilde{p}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)\tilde{p}_f^*(t) \tag{37}$$

where $\tilde{p}_f^*(t) = \hat{p}_f^*(t) - p_f^*(t)$. Similar to Theorem 3, for the Filippov solution of (36), one has $\dot{V}_3 \in^{a.e.} \dot{\tilde{V}}_3$ where

$$\begin{aligned}\dot{\tilde{V}}_3(t) &= \tilde{p}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)\dot{\hat{p}}_f^*(t) - \tilde{p}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)\dot{p}_f^*(t) \\ &= -\beta_2\tilde{p}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)K[\mathrm{sgn}]\left(\mathcal{L}_{ff}(\alpha^*)\tilde{p}_f^*(t)\right) \\ &+ \tilde{p}_f^{*\top}(t)\mathcal{L}_{fl}(\alpha^*)(1_{n_l} \otimes v_c^*).\end{aligned} \tag{38}$$

Using the facts that $\mathcal{L}_{fl}(\alpha^*)(1_{n_l} \otimes v_c^*) + \mathcal{L}_{ff}(\alpha^*)(1_{n_f} \otimes v_c^*) = 0$ Chen (2022), $x^{\top}K[\mathrm{sgn}](x) = \|x\|_1$ and $\|x\|_1 \ge \|x\|_2$ for an arbitrary vector $x = [x_1, \ldots, x_n]^{\top}$, one has

$$\begin{aligned}\dot{\tilde{V}}_3(t) &\le -(\beta_2 - \sqrt{n_f}\|v_c^*\|_2)\|\mathcal{L}_{ff}(\alpha^*)\tilde{p}_f^*(t)\|_2 \\ &\le -\sqrt{2}(\beta_2 - \sqrt{n_f}\|v_c^*\|_2)\frac{\lambda_{\min}(\mathcal{L}_{ff}(\alpha^*))}{\sqrt{\lambda_{\max}(\mathcal{L}_{ff}(\alpha^*))}}(V_3(t))^{\frac{1}{2}}\end{aligned} \tag{39}$$

where $\sqrt{2}(\beta_2 - \sqrt{n_f}\|v_c^*\|_2)\frac{\lambda_{\min}(\mathcal{L}_{ff}(\alpha^*))}{\sqrt{\lambda_{\max}(\mathcal{L}_{ff}(\alpha^*))}} > 0$. According to Hong et al. (2001, Lemma1), there exists a settling time $T_3 = \frac{\sqrt{2}\sqrt{\lambda_{\max}(\mathcal{L}_{ff}(\alpha^*))}}{(\beta_2 - \sqrt{n_f}\|v_c^*\|_2)\lambda_{\min}(\mathcal{L}_{ff}(\alpha^*))}V_3^{\frac{1}{2}}(0)$ such that for $\forall t > T_3$, $V_3(t) = 0$, i.e., $\hat{p}_f^*(t) = p_f^*(t)$. □

4.3. Simultaneous localization and formation

After the followers know their desired time-varying positions by (36), we now design a position estimator for each follower to obtain its real-time position in the global coordinate frame, and design a position-based formation control law such that each follower can move from its current position to its desired position. Different from the case of static leaders in Section 3, we design a SLAF algorithm for the followers as

$$\dot{\hat{p}}_f(t) = -\mathcal{L}_{ff}(\alpha(t))\hat{p}_f(t) - \mathcal{L}_{fl}(\alpha(t))p_l(t) + u_f(t) - k_2(\hat{p}_f(t) - \hat{p}_f^*(t)), \tag{40}$$

$$\dot{p}_f(t) = -k_2(\hat{p}_f(t) - \hat{p}_f^*(t)) - \int_0^t (\hat{p}_f(\tau) - \hat{p}_f^*(\tau))d\tau \tag{41}$$

where $k_2 > 1$ is a scalar. Similar to (18)–(19), one can also have the component form of the SLAF algorithm (40)–(41), from which one can see that each follower i needs the angle measurement information $\alpha_{kij}(t), \alpha_{ijk}(t), (i,j,k) \in \mathcal{A}$ and the communication information $\hat{p}_j(t), \forall j \in \mathcal{N}_i$. Therefore, the simultaneous localization and formation algorithm (40)–(41) is distributed. Since the states in (40)–(41) are bounded for $t \in [0, T_3]$, we conduct the stability analysis of the dynamics (40) and (41) for $t > T_3$. First, by defining an auxiliary variable $\tilde{\xi}_f(t) = \int_0^t(\hat{p}_f(\tau) - p_f^*(\tau))d\tau - \dot{p}_f^*(t)$, the closed-loop dynamics of (40) and (41) can be written as

$$\begin{bmatrix} \dot{\tilde{p}}_{ef}(t) \\ \dot{\tilde{p}}_f(t) \\ \dot{\tilde{\xi}}_f(t) \end{bmatrix} = -\begin{bmatrix} \mathcal{L}_{ff}(\alpha(t)) + k_2 I_{2n_f} & k_2 I_{2n_f} & 0 \\ k_2 I_{2n_f} & k_2 I_{2n_f} & I_{2n_f} \\ -I_{2n_f} & -I_{2n_f} & 0 \end{bmatrix} \begin{bmatrix} \tilde{p}_{ef}(t) \\ \tilde{p}_f(t) \\ \tilde{\xi}_f(t) \end{bmatrix} \tag{42}$$

where $\dot{p}_f^*(t) = -\mathcal{L}_{ff}^{-1}(\alpha^*)\mathcal{L}_{fl}(\alpha^*)\dot{p}_l^*(t) = -\mathcal{L}_{ff}^{-1}(\alpha^*)\mathcal{L}_{fl}(\alpha^*)(1_{n_l} \otimes v_c^*)$ is constant and we have used the fact $\ddot{p}_f^*(t) = 0$. Due to the existence of the integration term in (41), the SLAF dynamics (42) for the case of constant-velocity leaders is different from the SLAF dynamics (20) for the case of static leaders, even when the velocity v_c^* in (35) is zero. Now, we present the results about (42).

Theorem 5. *Consider a multi-agent system consisting of constant-velocity leaders and followers governed by (9). Suppose that each follower $i \in \mathcal{V}_f$ has angle measurements $\alpha_{jik}(t)$ and the information $\hat{p}_j^*(t), \hat{p}_j(t), \alpha_{ijk}(t)$ obtained from the communication with its neighbor $j \in \mathcal{N}_i$. Under the simultaneous localization and formation algorithm (40)–(41), if Assumption 1 holds and the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is localizable, then the following conclusions hold:*

(i) The equilibrium set of the dynamics (42) is

$$\begin{aligned} \Omega_{ed} &= \{(\tilde{p}_{ef}, \tilde{p}_f, \tilde{\xi}_f) |\ R_{\mathcal{A}}^f(\alpha)\tilde{p}_{ef} = 0, \tilde{p}_{ef} + \tilde{p}_f = 0, \tilde{\xi}_f = 0\} \\ &= \{(\tilde{p}_{ef}, \tilde{p}_f, \tilde{\xi}_f) |\ R_{\mathcal{A}}^f(\alpha)(p_f - p_f^*) = 0, \hat{p}_f = p_f^*, \tilde{\xi}_f = 0\}. \end{aligned}$$

(ii) The desired equilibrium $\{\tilde{p}_{ef} = 0, \tilde{p}_f = 0, \tilde{\xi}_f = 0\}$ of the dynamics (42) is locally and exponentially stable.

(iii) The desired equilibrium of the closed-loop dynamics (42) is asymptotically stable if for $\forall t > 0$,

$$(12k_2^2 - 8)\mathcal{L}_{ff}(\alpha(t)) - k_2^3\mathcal{L}_{ff}(\alpha(t))\mathcal{L}_{ff}(\alpha(t)) - 4k_2 I_{2n_f} > 0. \tag{43}$$

Proof of Theorem 5. (i): Letting $\dot{\tilde{\xi}}_f = 0$, one has $\tilde{p}_f + \tilde{p}_{ef} = 0$. Further letting $\dot{\tilde{p}}_{ef} = 0, \dot{\tilde{p}}_f = 0$, it follows that $\tilde{\xi}_f = 0$ and $\mathcal{L}_{ff}(\alpha)\tilde{p}_{ef} = 0$ which imply $R_{\mathcal{A}}^f(\alpha)\tilde{p}_{ef} = 0$. By using the definition of $\tilde{p}_{ef}$ and $\tilde{p}_f$, the equilibrium set of (42) can be written as Ω_{ed}. □

Proof of Theorem 5. (ii): Note that the closed-loop system (42) is an autonomous system. By following the linearization steps given in (26), one has that the linearized dynamics of (42) can be written as

$$\begin{bmatrix} \dot{\tilde{p}}_{ef}(t) \\ \dot{\tilde{p}}_f(t) \\ \dot{\tilde{\xi}}_f(t) \end{bmatrix} = -\begin{bmatrix} \mathcal{L}_{ff}(\alpha^*) + k_2 I_{2n_f} & k_2 I_{2n_f} & 0 \\ k_2 I_{2n_f} & k_2 I_{2n_f} & I_{2n_f} \\ -I_{2n_f} & -I_{2n_f} & 0 \end{bmatrix} \begin{bmatrix} \tilde{p}_{ef}(t) \\ \tilde{p}_f(t) \\ \tilde{\xi}_f(t) \end{bmatrix}. \tag{44}$$

Then, the characteristic polynomial of the system matrix of (44) can be written as

$$\begin{aligned} &\lambda^{-2n_f} \begin{vmatrix} \lambda(\lambda I_{2n_f} + \mathcal{L}_{ff}(\alpha^*) + k_2 I_{2n_f}) & \lambda k_2 I_{2n_f} \\ (k_2\lambda + 1)I_{2n_f} & (\lambda^2 + k_2\lambda + 1)I_{2n_f} \end{vmatrix} \\ &= \prod_{i=1}^{2n_f} \big(\lambda^3 + [2k_2 + \lambda_i(\mathcal{L}_{ff}(\alpha^*))]\lambda^2 \\ &\quad + [k_2\lambda_i(\mathcal{L}_{ff}(\alpha^*)) + 1]\lambda + \lambda_i(\mathcal{L}_{ff}(\alpha^*))\big) = 0 \end{aligned} \tag{45}$$

where $\mathcal{L}_{ff}(\alpha^*)$ is positive definite and can be written by $\mathcal{L}_{ff}(\alpha^*) = P_1 \text{diag}\{[\lambda_1(\mathcal{L}_{ff}(\alpha^*)), \ldots, \lambda_{2n_f}(\mathcal{L}_{ff}(\alpha^*))]\}P_1^{-1}$ where $P_1 \in \mathbb{R}^{2n_f \times 2n_f}$ is a nonsingular matrix. Using the checking condition for three-order characteristic polynomial (Chang & Chen, 1974), all the $6n_f$ solutions of (45) have negative real parts because $(2k_2 + \lambda_i(\mathcal{L}_{ff}(\alpha^*)))(k_2\lambda_i(\mathcal{L}_{ff}(\alpha^*)) + 1) - \lambda_i(\mathcal{L}_{ff}(\alpha^*)) > 0, \forall i = 1, \ldots, 2n_f$. Then, the equilibrium of the linearized system (44) is exponentially stable. When the initial state of (42) lies in a neighborhood of the desired equilibrium $\{\tilde{p}_{ef} = 0, \tilde{p}_f = 0, \tilde{\xi}_f = 0\}$, one has that the desired equilibrium of (42) is locally and exponentially stable by following the Lyapunov analysis in Khalil (2002, Theorem 4.7). □

Proof of Theorem 5. (iii): Construct the Lyapunov function candidate as

$$V_4(t) = X^\top(t)P_2X(t) \tag{46}$$

where $P_2 = \begin{bmatrix} I_{2n_f} & 0 & \frac{k_2 I_{2n_f}}{2} \\ 0 & I_{2n_f} & \frac{k_2 I_{2n_f}}{2} \\ \frac{k_2 I_{2n_f}}{2} & \frac{k_2 I_{2n_f}}{2} & k_2^2 I_{2n_f} \end{bmatrix}$, $X(t) = \begin{bmatrix} \tilde{p}_{ef}(t) \\ \tilde{p}_f(t) \\ \tilde{\xi}_f(t) \end{bmatrix}$. According to Schur complement theorem (Gallier et al., 2010), since $k_2^2 I_{2n_f} > 0$ and $\begin{bmatrix} I_{2n_f} & 0 \\ 0 & I_{2n_f} \end{bmatrix} - \begin{bmatrix} 0.25I_{2n_f} & 0.25I_{2n_f} \\ 0.25I_{2n_f} & 0.25I_{2n_f} \end{bmatrix} > 0$, one has that P_2 is symmetric and positive definite. It follows that $V_4(t)$ is positive definite. Taking the time-derivative of (46) along (42) yields

$$\dot{V}_3(t) = X^\top(t)(P_2A_1(t) + A_1^\top(t)P_2)X(t) = -X^\top(t)Q(t)X(t)$$

where $A_1(t)$ is the system matrix of (42), and

$$Q(t) = \begin{bmatrix} 2\mathcal{L}_{ff}(\alpha(t)) + k_2 I_{2n_f} & k_2 I_{2n_f} & \frac{k_2\mathcal{L}_{ff}(\alpha(t))}{2} \\ k_2 I_{2n_f} & k_2 I_{2n_f} & I_{2n_f} \\ \frac{k_2\mathcal{L}_{ff}(\alpha(t))}{2} & I_{2n_f} & k_2 I_{2n_f} \end{bmatrix}.$$

According to Lyapunov theorem for nonlinear autonomous systems, if $Q(t) > 0$, then the nonlinear dynamics (42) are globally stable. Since $k_2 I_{2n_f}$ is invertible and positive definite, one has that $Q(t) > 0$ iff

$$\begin{aligned} &\begin{bmatrix} 2\mathcal{L}_{ff}(\alpha(t)) + k_2 I_{2n_f} & k_2 I_{2n_f} \\ k_2 I_{2n_f} & k_2 I_{2n_f} \end{bmatrix} - \begin{bmatrix} \frac{k_2\mathcal{L}_{ff}(\alpha(t))\mathcal{L}_{ff}(\alpha(t))}{4} & \frac{\mathcal{L}_{ff}(\alpha(t))}{2} \\ \frac{\mathcal{L}_{ff}(\alpha(t))}{2} & 1/k_2 I_{2n_f} \end{bmatrix} \\ &= \begin{bmatrix} 2\mathcal{L}_{ff}(\alpha(t)) + k_2 I_{2n_f} - \frac{k_2\mathcal{L}_{ff}(\alpha(t))\mathcal{L}_{ff}(\alpha(t))}{4} & k_2 I_{2n_f} - \frac{\mathcal{L}_{ff}(\alpha(t))}{2} \\ k_2 I_{2n_f} - \frac{\mathcal{L}_{ff}(\alpha(t))}{2} & (k_2 - 1/k_2)I_{2n_f} \end{bmatrix} > 0 \end{aligned} \tag{47}$$

Note that since $(k_2 - 1/k_2)I_{2n_f}$ is invertible and positive definite, using Schur complement theorem (Gallier et al., 2010) again, one has that (47) holds iff

$$\begin{aligned} &2\mathcal{L}_{ff}(\alpha(t)) + k_2 I_{2n_f} - \frac{k_2\mathcal{L}_{ff}(\alpha(t))\mathcal{L}_{ff}(\alpha(t))}{4} \\ &> \frac{k_2(k_2 I_{2n_f} - \frac{\mathcal{L}_{ff}(\alpha(t))}{2})(k_2 I_{2n_f} - \frac{\mathcal{L}_{ff}(\alpha(t))}{2})}{k_2^2 - 1} \end{aligned} \tag{48}$$

which is equivalent to the condition (43). □

Remark 4. In the SLAF algorithm (40)–(41), the leaders' moving velocity v_c^* is neither directly known nor estimated by the followers. Therefore, (40)–(41) requires less global information and has less communication burden than some other existing formation or localization algorithms which need the knowledge of v_c^* or its distributed estimation (Guo et al., 2020). In addition, (43) is equivalent to requiring that $\forall y \in \mathbb{R}^{2n_f}$ with $\|y\| = 1$, $(12k_2^2 - 8)y^\top \mathcal{L}_{ff}(\alpha(t))y - k_2^3 y^\top \mathcal{L}_{ff}(\alpha(t))\mathcal{L}_{ff}(\alpha(t))y > 4k_2$.

5. SLAF for the case of leaders with time-varying moving velocities

Different from Section 4, the leaders in this section move with time-varying velocities, i.e., $\dot{p}_i^*(t) = v_i^*(t) \neq 0, t \geq 0, i \in \mathcal{V}_l$ where $v_i^*(t) \in \mathbb{R}^2$ is time-varying and is not necessarily equal to $v_j^*(t), i \neq j \in \mathcal{V}_l$.

5.1. Problem formulation

Consider the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*(t))$ is time-varying where $p^*(t)$ denotes the desired positions of all the agents. Note that the interior angles among the agents are invariant to the formation's translation, rotation and scaling. Therefore, if α^* are constant and $\mathbb{A}^*$ is localizable, $[p_l^{*\top}(t), p_f^{*\top}(t)]^\top$ must be the combination of the continuous translation, rotation, and scaling of $[p_l^{*\top}(0), p_f^{*\top}(0)]^\top$(Chen, 2022; Chen et al., 2021; Jing et al., 2019). More specifically,

$$p_l^*(t) = \kappa(t)[I_{n_l} \otimes \bar{R}(\theta(t))]p_l^*(0) + 1_{n_l} \otimes \mathcal{W}(t), t \geq 0$$

where $\kappa(t) \in \mathbb{R}$ is a nonzero scale factor, $\bar{R}(\theta(t)) \in SO(2)$ is a rotation matrix with rotation angle $\theta(t)$, $\mathcal{W}(t) \in \mathbb{R}^2$ is a translation vector, and $\kappa(t), \bar{R}(\theta(t)), \mathcal{W}(t)$ are all continuous and differential functions. Therefore, one can properly select $\mathcal{W}(t), \theta(t)$, and $\kappa(t)$ to perform desired translational, rotational and scaling formation maneuvering, respectively. For example, selecting $\theta(t) = \theta(0) + \omega_1 t$ and constant $\omega_1, \kappa, \mathcal{W}$ will give circular formation motion, while selecting $\mathcal{W}(t) = \mathcal{W}(0) + \begin{bmatrix} a_1 \cos \omega_2 t \\ b_1 \sin \omega_2 t \end{bmatrix}$, nonzero $|a_1| \neq |b_1|$ and constant ω_2, κ, θ will give elliptical formation motion. First, we give an assumption on the leaders' moving velocities $v_l^*(t) = \dot{p}_l^*$ and acceleration $\dot{v}_l^*(t), i \in \mathcal{V}_l$.

Assumption 2. The leaders' desired positions $p_l^*(t)$ are continuous and twice differentiable, and $\dot{p}_l^*(t) = v_l^*(t)$ is bounded by $\|v_l^*(t)\| \leq v_{\max}$, and $\dot{v}_l^*(t)$ is bounded by $\|\dot{v}_l^*(t)\| \leq a_{\max}$ where $v_{\max}, a_{\max}$ are positive constants and known by the followers.

5.2. Estimation of the followers' desired positions and velocities

Instead of only estimating the desired positions, the desired moving velocities also need to be estimated when the leaders move with time-varying velocities $v_l^*(t) = [v_1^{*\top}(t), \ldots, v_{n_l}^{*\top}(t)]^\top \in \mathbb{R}^{2n_l}$. We design desired position and velocity estimators as

$$\dot{\hat{p}}_f^*(t) = -\beta_3 \mathrm{sgn}\left(\mathcal{L}_{ff}(\alpha^*)\hat{p}_f^*(t) + \mathcal{L}_{fl}(\alpha^*)p_l^*(t)\right), \tag{49}$$

$$\dot{\hat{v}}_f^*(t) = -\beta_4 \mathrm{sgn}\left(\mathcal{L}_{ff}(\alpha^*)\hat{v}_f^*(t) + \mathcal{L}_{fl}(\alpha^*)v_l^*(t)\right) \tag{50}$$

where $\beta_3 > \|\mathcal{L}_{ff}^{-1}(\alpha^*)\mathcal{L}_{fl}(\alpha^*)\|v_{\max} \geq \|v_f^*(t)\|$, $\beta_4 > \|\mathcal{L}_{ff}^{-1}(\alpha^*)\mathcal{L}_{fl}(\alpha^*)\|a_{\max} \geq \|\dot{v}_f^*(t)\|$, $v_f^*(t) = -\mathcal{L}_{ff}^{-1}(\alpha^*)\mathcal{L}_{fl}(\alpha^*)v_l^*(t) = [v_{n_l+1}^{*\top}(t), \ldots, v_n^{*\top}(t)]^\top \in \mathbb{R}^{2n_f}$ represents the followers' desired velocities, and the component forms of (49)–(50) can be similarly obtained by following (13). The designed position estimation law (49) needs the desired angle information $\alpha_{ijk}^*, \alpha_{kij}^*$, the communication information $\hat{p}_j^*(t), j \in \mathcal{N}_i$ from its neighbors, and the upper bound of the desired moving velocity $v_f^*(t)$.

Theorem 6. *If the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*(t))$ is localizable, then $\hat{p}_f^*(t)$ and $\hat{v}_f^*(t)$ under (49)–(50) converge to $p_f^*(t)$ and $v_f^*(t)$, respectively, within finite time.*

Proof. For the estimation of $\hat{p}_f^*(t)$ in (49), note that (37)–(38) still holds. It follows that in this case one has

$$\begin{aligned}\dot{\tilde{V}}_3(t) &= -\beta_3\|\mathcal{L}_{ff}(\alpha^*)\tilde{p}_f^*(t)\|_1 + \tilde{p}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)v_f^*(t) \\ &\leq -(\beta_3 - \|v_f^*(t)\|)\|\mathcal{L}_{ff}(\alpha^*)\tilde{p}_f^*(t)\|_2.\end{aligned}$$

Following (39), $\tilde{p}_f^*(t)$ converges to zero within finite time. For the estimation of $\hat{v}_f^*(t)$, design the Lyapunov function candidate as $V_5(t) = 0.5\tilde{v}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)\tilde{v}_f^*(t)$ where $\tilde{v}_f^*(t) = \hat{v}_f^*(t) - v_f^*(t)$. Similar to (38) and (39), for the Filippov solution of (50), one has $\dot{V}_5 \in^{a.e.} \dot{\tilde{V}}_5$ where $\dot{\tilde{V}}_5(t) = -\beta_4\|\mathcal{L}_{ff}(\alpha^*)\tilde{v}_f^*(t)\|_1 + \tilde{v}_f^{*\top}(t)\mathcal{L}_{ff}(\alpha^*)\dot{v}_f^*(t) \leq -\sqrt{2}(\beta_4 - ||\dot{v}_f^*(t)||)\frac{\lambda_{\min}(\mathcal{L}_{ff}(\alpha^*))}{\sqrt{\lambda_{\max}(\mathcal{L}_{ff}(\alpha^*))}}(V_5(t))^{\frac{1}{2}}$ which implies that $\tilde{v}_f^*(t)$ converges to zero within finite time. □

5.3. Simultaneous localization and formation

After having the knowledge of $p_f^*(t), v_f^*(t)$, we now design position estimator $\dot{\hat{p}}_f(t)$ to localize the followers in the global coordinate frame, and design formation control law $u_f(t)$ such that the followers converge to their desired time-varying positions. Towards this end, we design a SLAF algorithm as

$$\begin{aligned}\dot{\hat{p}}_f(t) = &- \mathcal{L}_{ff}(\alpha(t))\hat{p}_f(t) - \mathcal{L}_{fl}(\alpha(t))p_l(t) + u_f(t) \\ &- k_1(\hat{p}_f(t) - \hat{p}_f^*(t)) - u_{aux}(t),\end{aligned} \tag{51}$$

$$u_f(t) = \hat{v}_f^*(t) - k_1(\hat{p}_f(t) - \hat{p}_f^*(t)) - u_{aux}(t), \tag{52}$$

where $u_{aux}(t) = \mathrm{diag}\{[e_{n_l+1}(t)I_2, \ldots, e_n(t)I_2]\}(\mathrm{sgn}(\hat{p}_f(t) - p_f^*(t)) - \delta_f(t)) \in \mathbb{R}^{2n_f}$, and $\delta_f = [\delta_{n_l+1}^\top(t), \ldots, \delta_n^\top(t)]^\top \in \mathbb{R}^{2n_f}$, and $e_i, \delta_i, i \in \mathcal{V}_f$ have the same definitions as those in (31)–(32). Different from (41), the followers' formation control law (52) has a feedforward term $\hat{v}_f^*(t)$. Since the states in (51)–(52) are bounded within the finite time required by the convergence of (49)–(50), we can similarly replace $\hat{v}_f^*, \hat{p}_f^*$ by v_f^*, p_f^*, respectively after the finite time. Then, the component form of the SLAF algorithm (51) and (52) for each follower $i \in \mathcal{V}_f$ can be written as

$$\begin{aligned}\dot{\hat{p}}_i(t) = &- \sum_{(i,j_1,k_1)\in\bar{\mathcal{A}}} (A_i^{\triangle ij_1k_1}(\alpha(t)))^\top f_i^{\triangle ij_1k_1}(\alpha(t), \hat{p}(t)) \\ &- \sum_{(j_2,i,k_2)\in\bar{\mathcal{A}}} (A_i^{\triangle j_2ik_2}(\alpha(t)))^\top f_i^{\triangle j_2ik_2}(\alpha(t), \hat{p}(t)) \\ &- \sum_{(j_3,k_3,i)\in\bar{\mathcal{A}}} (A_i^{\triangle j_3k_3i}(\alpha(t)))^\top f_i^{\triangle j_3k_3i}(\alpha(t), \hat{p}(t)) \\ &+ u_i(t) - k_1(\hat{p}_i(t) - p_i^*(t)) \\ &- e_i(t)\left(\mathrm{sgn}(\hat{p}_i(t) - p_i^*(t)) - \delta_i(t)\right),\end{aligned} \tag{53}$$

$$\begin{aligned}\dot{p}_i(t) = u_i(t) = &\, v_i^*(t) - k_1(\hat{p}_i(t) - p_i^*(t)) \\ &- e_i(t)\left(\mathrm{sgn}(\hat{p}_i(t) - p_i^*(t)) - \delta_i(t)\right)\end{aligned} \tag{54}$$

where we have used $\hat{p}_i^* = p_i^*, \hat{v}_i^* = v_i^*$. Now, we give the results about the stability of (51)–(52).

Theorem 7. *Consider a multi-agent system consisting of leaders with time-varying velocities $v_l^*(t)$ and followers governed by (9). Suppose that each follower $i \in \mathcal{V}_f$ has angle measurements $\alpha_{jik}(t)$ and the information $\hat{p}_j^*(t), \hat{v}_j^*(t), \hat{p}_j(t), \alpha_{ijk}(t)$ obtained from*

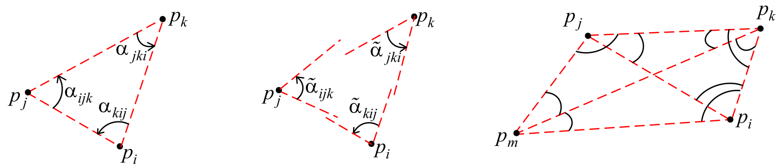


Fig. 2. Noiseless, noisy and no feasible solution cases.

the communication with its neighbors $j \in \mathcal{N}_i$. Under the simultaneous localization and formation algorithm (51)–(52), if Assumption 1 holds and $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is localizable, then the closed-loop system asymptotically converges to the desired equilibrium Ω_{es1}.

Proof. Similar to Theorem 3, one has that Ω_{es1} is the only equilibrium of (51)–(52). Using the same Lyapunov functions $V_2(t)$ and $\tilde{V}_2(t)$ for (51)–(52), one has

$$\begin{aligned}\dot{\tilde{V}}_2(t) &= -\tilde{p}_{ef}^\top(t)\mathcal{L}_{ff}(\alpha(t))\tilde{p}_{ef}(t) - k_1\|\tilde{p}_{ef}(t) + \tilde{p}_f(t)\|^2 \\ &- \sum_{i=n_a+1}^{n_a+n_f} e_i(t)(\hat{p}_i(t) - p_i(t))^\top \left(\mathrm{sgn}(\hat{p}_i(t) - p_i^*(t)) - \delta_i(t)\right) \\ &- \sum_{i=n_a+1}^{n_a+n_f} e_i(t)\left(p_i(t) - p_i^*(t)\right)^\top(\mathrm{sgn}(\hat{p}_i(t) - p_i^*(t)) - \delta_i(t)) \\ &\le -\tilde{p}_{ef}^\top(t)\mathcal{L}_{ff}(\alpha(t))\tilde{p}_{ef}(t) - k_1\|\tilde{p}_{ef}(t) + \tilde{p}_f(t)\|^2 \le 0\end{aligned}$$

which implies that the solution of (51)–(52) converges to Ω_{es}. Following the proof of Theorem 3 for nonsmooth system (51)–(52), the closed-loop system asymptotically converges to Ω_{es1}. □

Remark 5. When the leaders have time-varying velocities, an additional velocity estimator $\hat{v}_f^*(t)$ is designed in (50), which needs more communication resources and computations than the SLAF algorithm (40)–(41) designed for constant-velocity leaders and (16)–(17) designed for static leaders. Therefore, the SLAF algorithm (40)–(41) is not the special case of (51)–(52) or general case of (16)–(17). Due to the existence of an integral term in (41), even by adding a perturbation into (40)–(41), we have not proved its asymptotic stability. Therefore, compared to (40)–(41), the advantage of (51)–(52) and (16)–(17) is the asymptotic convergence. Although the information of n_f, $v_{\max}$, $a_{\max}$ and $\mathcal{L}(\alpha^*)$ is required in the estimation laws (36),(49),(50), they are usually known in the design stage or one can select sufficiently large β_i, $i = 2, 3, 4$ such that the corresponding conditions hold.

6. Further discussion on sensor measurements and convergence properties

This section investigates the effect of angle measurement noises, the extension to other sensor measurements, requirement on orientations of agents' coordinate frames, collision and collinearity avoidance. We only focus on the case that the leaders are static since the cases that the leaders are dynamic can be similarly obtained.

6.1. Effect of sensor measurement noises

Firstly, we consider that angles are measured by cameras. According to (1), the angle $\alpha_{kij}(t)$ is calculated by using $b_{ij}(t) = \begin{bmatrix}\cos\phi_{ij}(t) \\ \sin\phi_{ij}(t)\end{bmatrix}$ and $b_{ik} = \begin{bmatrix}\cos\phi_{ik}(t) \\ \sin\phi_{ik}(t)\end{bmatrix}$ where $\phi_{ij}(t) \in [0, 2\pi)$ is the bearing angle rotating from the X-axis of agent i's coordinate frame to the bearing direction $\overrightarrow{ij}$ under the counterclockwise direction. Assume that agent i's all bearing angle measurements are obtained from its onboard camera's images which subject to the same additive noise $\omega_3(t) \in \mathbb{R}$. Then, those noisy bearings can be described by $\tilde{b}_{ij}(t) = \begin{bmatrix}\cos(\phi_{ij}(t)+\omega_3(t)) \\ \sin(\phi_{ij}(t)+\omega_3(t))\end{bmatrix}$ and $\tilde{b}_{ik}(t) = \begin{bmatrix}\cos(\phi_{ik}(t)+\omega_3(t)) \\ \sin(\phi_{ik}(t)+\omega_3(t))\end{bmatrix}$. Since the matrices $\mathcal{L}_{ff}(\alpha(t))$, $\mathcal{L}_{fl}(\alpha(t))$ in (16) are only related to the interior angles $\alpha(t)$, we need to analyze the change of these matrices under the existence of noise $\omega_3(t)$. Defining $\tilde{\alpha}_{kij}(t)$ as the angle that is calculated by noisy $\tilde{b}_{ik}(t)$, $\tilde{b}_{ij}(t)$, it can be verified that $\tilde{\alpha}_{kij}(t) = \alpha_{kij}(t)$, which implies that the existence of $\omega_3(t)$ has no effect on $\alpha(t)$, and thus has no effect on $\mathcal{L}_{ff}(\alpha(t))$, $\mathcal{L}_{fl}(\alpha(t))$ and the proposed SLAF algorithms.

Secondly, we consider that angles are measured by directional antenna arrays, under which bearing angle measurements ϕ_{ij} and ϕ_{ik} subject to different additive noises, i.e., $\tilde{\alpha}_{kij}(t) \neq \alpha_{kij}(t)$. According to Fig. 2(a) and the angle-induced linear Eq. (3) in $\triangle ijk$, if p_i, p_j are known, p_k can be uniquely localized as $p_k = \frac{\bar{R}^\top(\alpha_{kij})}{\sin\alpha_{jki}}[\left(\sin\alpha_{jki}\bar{R}(\alpha_{kij}) - \sin\alpha_{ijk}I_2\right)p_i + \sin\alpha_{ijk}p_j]$. However, if measurement noises exist in the angle measurements $\alpha_{jki}, \alpha_{kij}, \alpha_{ijk}$, then the true position p_k calculated from noiseless angle measurements is different from $\tilde{p}_k = \frac{\bar{R}^\top(\tilde{\alpha}_{kij})}{\sin\tilde{\alpha}_{jki}}[\left(\sin\tilde{\alpha}_{jki}\bar{R}(\tilde{\alpha}_{kij}) - \sin\tilde{\alpha}_{ijk}I_2\right)p_i + \sin\tilde{\alpha}_{ijk}p_j]$ obtained from noisy angle measurements, see Fig. 2(b). Worse still, as shown in Fig. 2(c), if p_i, p_j are known and all the angle measurements are noisy, then there is no feasible solution for p_k, p_m under the angle measurements in triangles $\triangle ijk, \triangle jmk, \triangle imk$ (this is because there are 2 unknown position variables p_k, p_m, but 3 angle-induced linear equations). In this case, there does not exist an equilibrium for the four agents to achieve a desired or biased formation.

Therefore, to guarantee that the localization process under the existence of noises has at least a feasible solution, we must ensure $\bar{\alpha}_{jki}(\tilde{\alpha}_{jki}, \tilde{\alpha}_{kij}, \tilde{\alpha}_{ijk}) + \bar{\alpha}_{kij}(\tilde{\alpha}_{jki}, \tilde{\alpha}_{kij}, \tilde{\alpha}_{ijk}) + \bar{\alpha}_{ijk}(\tilde{\alpha}_{jki}, \tilde{\alpha}_{kij}, \tilde{\alpha}_{ijk}) = \pi$ (only discuss the case $\alpha_{jki} \in (0, \pi)$) where $\bar{\alpha}_{jki}(\tilde{\alpha}_{jki}, \tilde{\alpha}_{kij}, \tilde{\alpha}_{ijk})$ represents the angle that agent k will use for localization, which corresponds to α_{jki} and is a function of the measured noisy angles $\tilde{\alpha}_{jki}, \tilde{\alpha}_{kij}, \tilde{\alpha}_{ijk}$. To also minimize the negative effect of measurement noises, according to Lin et al. (2020), one can calculate $\bar{\alpha}_{kij}$ by using $\bar{\alpha}_{kij} = \tilde{\alpha}_{kij} - \frac{\pi - (\tilde{\alpha}_{jki} + \tilde{\alpha}_{kij} + \tilde{\alpha}_{ijk})}{3}$. Then, if p_i, p_j are known, there at least exists $\bar{p}_k$ such that $\left(\sin\bar{\alpha}_{jki}\bar{R}(\bar{\alpha}_{kij}) - \sin\bar{\alpha}_{ijk}I_2\right)p_i + \sin\bar{\alpha}_{ijk}p_j - \sin\bar{\alpha}_{jki}\bar{R}(\bar{\alpha}_{kij})\bar{p}_k = 0$. Moreover, if noises are bounded and sufficiently small (so that the angles $\bar{\alpha}_{jki}$ are still generic, i.e., not being 0 and π), then $\|p_k - \bar{p}_k\|$ should be bounded, i.e., there exists at least a biased formation for the agents to achieve. Therefore, for a network with multiple triangles, if more angle measurements are given, the negative effect of noises can be lessened by using some geometric properties existing in the network.

6.2. Extension to other types of sensor measurements

We extend the designed SLAF algorithms to the cases with distance or bearing measurements by establishing their corresponding measurement-induced linear equations.

(a) Distance measurements: Let $d_{ij} := \|p_j - p_i\|$ be the distance between agents i and j. According to Diao et al. (2014), the barycentric coordinates can be used to establish a distance-induced linear equation among four agents. To avoid singularities of the barycentric coordinates at collinear or unlocalizable configurations, we use barycentric-modified matrices to establish a distance-induced linear equation in quadrilateral $\varoslash ijkh$

$$f^d_{lijk}(d,p) = \mathcal{Q}_l^{\varoslash ijkh}(d)p_l + \mathcal{Q}_i^{\varoslash ijkh}(d)p_i + \mathcal{Q}_j^{\varoslash ijkh}(d)p_j + \mathcal{Q}_k^{\varoslash ijkh}(d)p_k = 0 \tag{55}$$

where $\mathcal{Q}_l^{\varoslash ijkh}(d) = S_{\triangle ijk}I_2 \in \mathbb{R}^{2\times 2}$, $\mathcal{Q}_i^{\varoslash ijkh}(d) = -S_{\triangle ljk}I_2$, $\mathcal{Q}_j^{\varoslash ijkh}(d) = -S_{\triangle lki}I_2$, $\mathcal{Q}_k^{\varoslash ijkh}(d) = -S_{\triangle lij}I_2$, and $S_{\triangle ljk}$, $S_{\triangle lki}$, $S_{\triangle lij}$, $S_{\triangle ijk}$ are the signed area of the corresponding triangles $\triangle ljk, \triangle lki, \triangle lij, \triangle ijk$, respectively. The value of $S_{\triangle ljk}, S_{\triangle lki}, S_{\triangle lij}$, $S_{\triangle ijk}$ can be calculated by Cayley–Menger determinant and sign determination formulas in Diao et al. (2014), which are only related to inter-node distances. If three points of p_i, p_j, p_k, p_l are collinear, (55) is still well-defined but the remaining non-collinear point will not be shown in (55).

(b) Bearing measurements: According to Zhao and Zelazo (2016), the bearing-induced linear equation between agents i and j can be described as

$$f^b_{ij}(b,p) = (I_2 - b_{ij}b_{ij}^{\top})(p_i - p_j) = 0 \tag{56}$$

Writing all the induced linear Eqs. (55) or (56) from the multi-agent network into a compact form, the corresponding measurement and localization matrices can be similarly defined for network localizability and localization. Also, SLAF algorithms in Sections 3, 4, 5 can be similarly developed.

6.3. Requirement on orientations of agents' coordinate frames

Consider that each agent holds a sensor measurement coordinate frame and a control execution coordinate frame. The proposed SLAF strategies require the orientations of the agents' control execution coordinate frames to be aligned with the global coordinate frame, but has no requirement on the orientations of the agents' sensing coordinate frames. Moreover, given the available angle measurements and wireless communication among the agents, the relative orientation from each agent's local coordinate frame to the global coordinate frame can be approximately determined at the initial stage, which can be used to align agents' local coordinate frames with the global coordinate frame. More specifically, at the initial stage where all the agents are static, by running the angle-only localization algorithm (8), agent i's estimated position $\hat{p}_i$ will converge to its absolute position $p_i(0)$ in the global coordinate frame $\sum_g$ at an exponential convergent rate (assume that the agents' initial configuration is localizable). Then, by communicating with agent $k \in \mathcal{N}_i$, agent i knows the estimated relative position $\hat{p}_{ik} = \hat{p}_k - \hat{p}_i$ which exponentially converges to $p_{ik}(0)$. At the same time, if agent i can measure the relative bearing b^i_{ik} with respect to agent k in its local coordinate frame $\sum_i$, then the relative orientation $R^g_i(\theta) \in SO(2)$ from $\sum_i$ to $\sum_g$ can be approximately determined by using the fact $p_{ik}(0)/\|p_{ik}(0)\| = b_{ik}(0) \approx \hat{b}_{ik} = \frac{\hat{p}_k - \hat{p}_i}{\|\hat{p}_k - \hat{p}_i\|} = R^g_i(\theta)b^i_{ik}$.

6.4. Collision avoidance

Now, we relax Assumption 1 by constraining agents' initial states. Suppose that the initial formation is close to the target formation. For any two agents $i, j \in \mathcal{V}$ in Section 3, one has $p_i(t) - p_j(t) = p_i(t) - p_i^* - (p_j(t) - p_j^*) + p_i^* - p_j^*$. According to Zhao et al. (2019, Eq. (16)), one has

$$\begin{aligned}\|p_i(t) - p_j(t)\| &\geq \|p_i^* - p_j^*\| - \|p_i(t) - p_i^*\| - \|p_j(t) - p_j^*\| \\ &\geq \|p_i^* - p_j^*\| - \sqrt{n_f}\|Y(t)\|\end{aligned} \tag{57}$$

We consider the evolution of (20) is from $t = 0$, which can be fulfilled by running (12) for more than T_1 seconds at the initial stage.

Corollary 1. *Suppose that γ is the desired minimum separation between any two agents during the formation evolution of (20) and satisfies $0 < \gamma < \min_{i,j\in\mathcal{V}} \|p_i^* - p_j^*\|$. If $\|Y(0)\| \leq \varepsilon := \frac{1}{\sqrt{n_f}}\left(\min_{i,j\in\mathcal{V}} \|p_i^* - p_j^*\| - \gamma\right)$, then $\|p_i(t) - p_j(t)\| \geq \gamma$, for all $t > 0$.*

Proof. For $t = 0$, substituting the condition on γ into (57) yields $\|p_i(0) - p_j(0)\| \geq \gamma > 0$ which implies no collision among agents at the initial time. Suppose the first case that no collision occurs among agents for $t > 0$. Since $\dot{V}_2(t) \leq 0$ as shown in (25), according to (57), one has $\|p_i(t) - p_j(t)\| \geq \|p_i^* - p_j^*\| - \sqrt{n_f}\|Y(0)\| \geq \gamma$. Suppose the second case that there exists a collision at $t = T_4$ between agents i and j during the evolution. Then, there must exist an escape time $T_5 > 0$ such that $0 < T_5 \leq T_4^-$, $\|Y(T_5)\| = \varepsilon$ and $\dot{V}_2(T_5) > 0$. This contradicts with the fact $\dot{V}_2(t) \leq 0$ for $t \in [0, T_4^-]$. As a result, $\|p_i(t) - p_j(t)\| \geq \gamma$ holds $\forall t > 0$. □

The above conclusion also holds for Section 3.4, and Section 5.3 if one replaces $p_i^* - p_j^*$ with $p_i^*(t) - p_j^*(t)$ in (57).

6.5. Collinearity avoidance

In Theorem 2. (iii), the asymptotic stability is obtained if no collinearity among neighboring agents occurs. Now, we aim to avoid collinearity by constraining agents' initial states. Note that

$$p_i(t) - p_i(0) = p_i(t) - p_i^* - (p_i(0) - p_i^*) \tag{58}$$

Considering the evolution of (20) also from $t = 0$, if there is no collision $\forall t \geq 0$ (can be guaranteed by Corollary 1), (20) has a solution. Then, one has

$$\begin{aligned}&\|p_i(t) - p_i(0)\| \leq \|p_i(0) - p_i^*\| + \|p_i(t) - p_i^*\| \\ &\leq \|p_i(0) - p_i^*\| + \sqrt{n_f}\|\tilde{p}_f(t)\| \leq \|p_i(0) - p_i^*\| + \sqrt{n_f}\|Y(t)\| \\ &\leq \|p_i(0) - p_i^*\| + \sqrt{n_f}\|Y(0)\|\end{aligned} \tag{59}$$

where we have used the facts that $\dot{V}_2(t) \leq 0$ and $\|Y(t)\|$ is non-increasing. Then, each agent i's movement from $t = 0$ to $+\infty$ is within the circular region $\mathcal{R}_i = \{p_i \in \mathbb{R}^2 | \; \|p_i - p_i(0)\| \leq \|p_i(0) - p_i^*\| + \sqrt{n_f}\|Y(0)\|\}$. Now, we discuss the collinearity issue within $\triangle ijk, (i,j,k) \in \mathcal{A}$. To get a sufficient condition guaranteeing no collinearity among i, j, k, we first draw the region $\mathcal{R}_{ij}$ that $\overleftrightarrow{p_i(t)p_j(t)}$ will cover, where we require $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ (otherwise a collinearity already occurs). As shown in Fig. 3, for two circular regions $\mathcal{R}_i, \mathcal{R}_j$, we can draw two external tangent lines $\overleftrightarrow{i'j'}$, $\overleftrightarrow{i''j''}$, and also two internal tangent lines $\overleftrightarrow{i'''j'''}$, $\overleftrightarrow{i''''j''''}$. Denote by I_1, I_3, J_1, J_3 the intersections of these four tangent lines. Denote by I_2 a point that lies in the ray $\overrightarrow{J_3I_1}$ but $I_2 \notin \overline{J_3I_1}$. The same definition applies for I_4, J_2, J_4, see Fig. 3. Then, $\mathcal{R}_{ij}$ represents the open region between the border $J_2J_1I_1I_2$ and the border $J_4J_3I_3I_4$. It can be verified that if $\mathcal{R}_{ij} \cap \mathcal{R}_k = \emptyset$, then three arbitrary points lying in these three circular regions $\mathcal{R}_i, \mathcal{R}_j, \mathcal{R}_k$, respectively, will not be collinear, i.e., no collinearity will occur among i, j, k. Using similar definitions, if $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$ and $\mathcal{R}_{ij} \cap \mathcal{R}_k = \emptyset$ holds for every $(i,j,k) \in \mathcal{A}$, then no collinearity will occur among all the neighboring agents. Combining the collision analysis in Section 6.4 and the collinearity analysis in this section, the set $\mathbb{U}$ mentioned in the proof of Theorem 2. (ii) can be described by $\mathbb{U} = \{Y(0) \big| \|Y(0)\| \leq \frac{\min_{i,j\in\mathcal{V}} \|p_i^* - p_j^*\|}{2\sqrt{n_f}}, \mathcal{R}_i \cap \mathcal{R}_j = \emptyset, \mathcal{R}_{ij} \cap \mathcal{R}_k = \emptyset, \forall (i,j,k) \in \mathcal{A}\}$, where $\mathcal{R}_i, \mathcal{R}_j, \mathcal{R}_k, \mathcal{R}_{ij}$ are functions of $Y(0)$ and

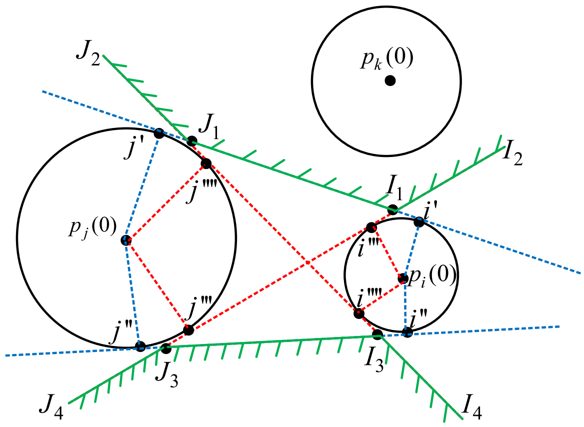


Fig. 3. Geometric interpretation for collinearity avoidance.

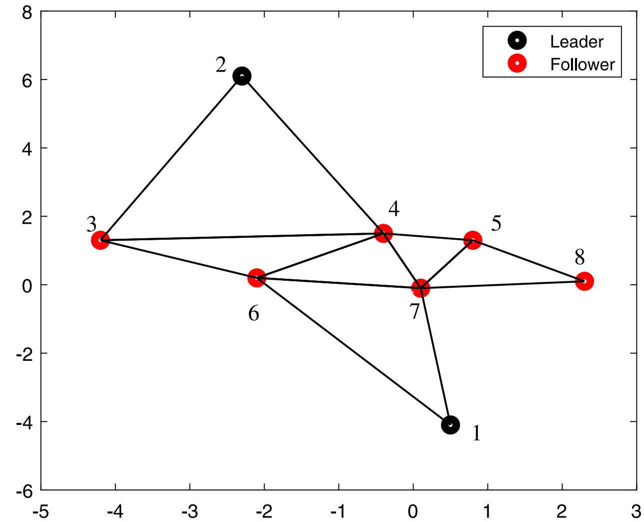


Fig. 4. Network topology among 2 leaders and 6 followers.

we selected $\gamma = \frac{\min_{i,j\in\mathcal{V}} \|p_i^*-p_j^*\|}{2\sqrt{n_f}}$. Since the radius of $\mathcal{R}_i$, $\mathcal{R}_j$ and $\mathcal{R}_k$ is proportional to $\|Y(0)\|$ according to (59), the above described $\mathbb{U}$ is indeed a small neighborhood of the equilibrium $Y = 0$.

7. Simulation examples

This section presents four simulation examples to validate Theorems 2, 3, 5, 7 respectively. The multi-agent network is shown in Fig. 4, which contains 6 triangles: $\triangle 234$, $\triangle 346$, $\triangle 467$, $\triangle 457$, $\triangle 578$, $\triangle 167$. Note that the two leaders are non-neighboring, which is designed in this way to demonstrate the advantages of the SLAF algorithms over sequential formations or localization algorithms which require leaders to be neighbors to one another (Chen et al., 2021; Jing et al., 2021). The initial positions of the agents are: $p_1(0) = [0.5; -4.1]$, $p_2(0) = [-2.3; 6.1]$, $p_3(0) = [-18.84; -11.04]$, $p_4(0) = [-15.6; -13.2]$, $p_5(0) = [-12.84; -12.6]$, $p_6(0) = [-18; -13.92]$, $p_7(0) = [-14.88; -12.48]$, $p_8(0) = [-15.36; -11.28]$. For the case of static leaders, the followers' desired positions are $p_3^* = [-4.2; 1.3]$, $p_4^* = [-0.4; 1.5]$, $p_5^* = [0.8; 1.3]$, $p_6^* = [-2.1; 0.2]$, $p_7^* = [0.1; -0.1]$, $p_8 = [2.3; 0.1]$. The initial estimates $\hat{p}_f(0) = 1.5p_f(0)$, and $k_1 = k_2 = 10$. The video showing dynamical formation trajectories of these four cases is uploaded to https://youtu.be/r5PbTUJ7UzE.

7.1. Case 1: SLAF when leaders are static

Under the SLAF law (16)–(17), Figs. 5–6 show the formation errors and trajectories where a collinearity occurs. Note that if the initial formation is close to the desired formation, no collinearity usually will occur. To show possibilities of collinearity at the evolution of the formation, the initial formation is far away from the desired formation in this case.

7.2. Case 2: Perturbation-based SLAF when leaders are static

Under the SLAF law (31)–(32), Figs. 7–8 show the formation errors and trajectories where the collinearity is avoided and the perturbation in the control law is chosen as $\delta_i = [0.5\cos(200\pi t); 0.5\sin(200\pi t)]$, $i = 3, \ldots, 8$. Due to the convergence of the angle errors, the effect of the added perturbation on the convergence of formation errors is limited.

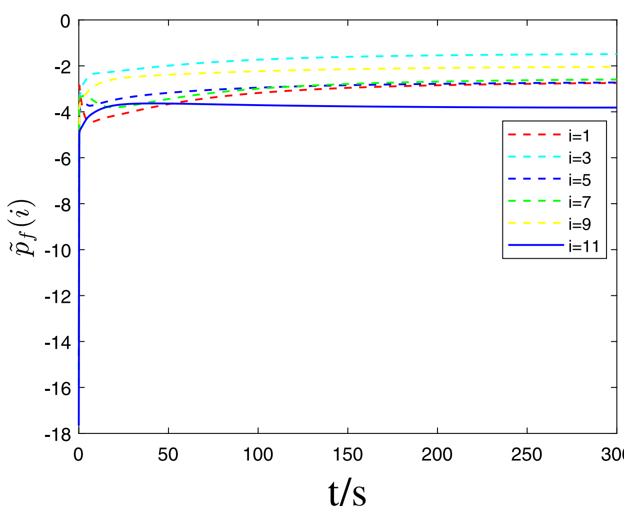


Fig. 5. Formation errors in Case 1.

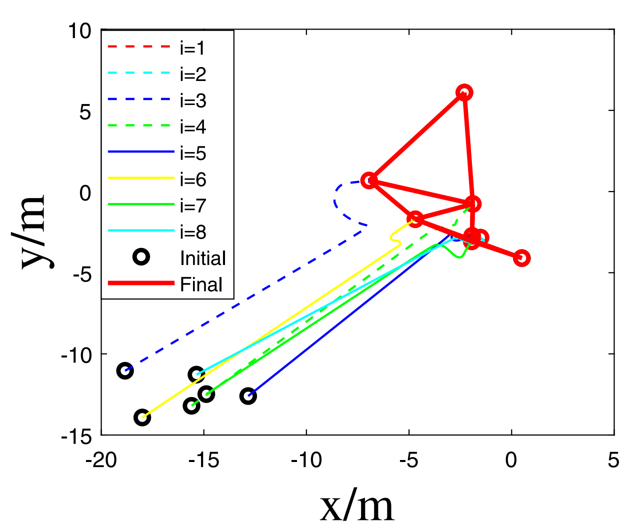


Fig. 6. Formation trajectories in Case 1.

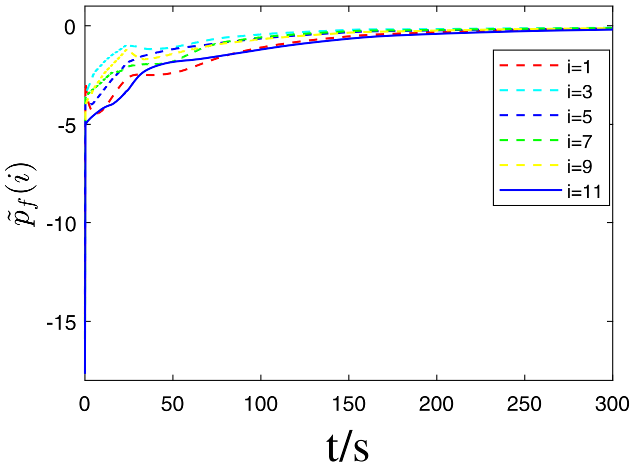


Fig. 7. Formation errors in Case 2.

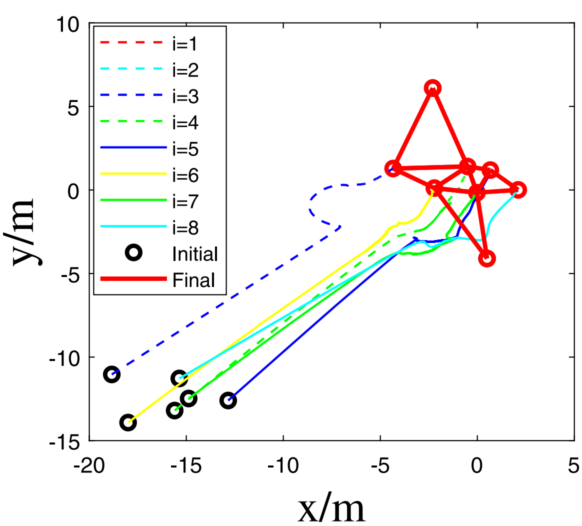


Fig. 8. Formation trajectories in Case 2.

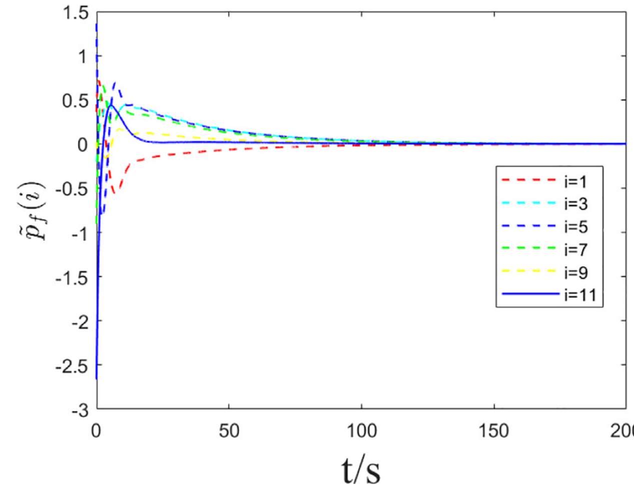


Fig. 9. Formation errors in Case 3.

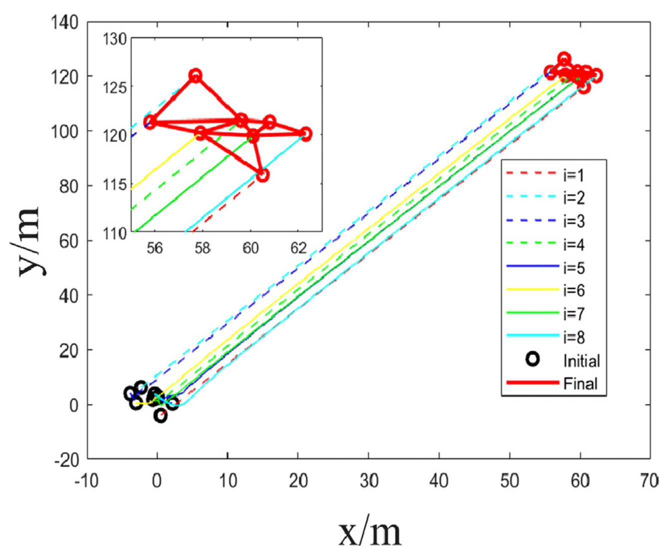


Fig. 10. Formation trajectories in Case 3.

7.3. Case 3: SLAF when leaders move with constant velocity

Under the SLAF law (40)–(41), the formation errors and trajectories are given in Figs. 9–10, where the desired formation is achieved and the formation finally translates at the desired velocity. Note that the convergence speed in this case is not very fast. This is because the desired moving velocity is unknown for all the followers and they can only use the integration term in the control law (41) to compensate for the formation errors, which usually costs some time.

7.4. Case 4: SLAF when leaders move with time-varying velocity

First, we simulate the SLAF algorithm (51)–(52) without perturbation, i.e., letting $u_{aux} = 0$. The formation errors and trajectories are given in Figs. 11 and 13, in which a collinearity occurs. Then, we simulate (51)–(52) with perturbation term, i.e., $u_{aux} \neq 0$. The formation errors and trajectories are given in Figs. 12 and 14, in which the collinearity is avoided.

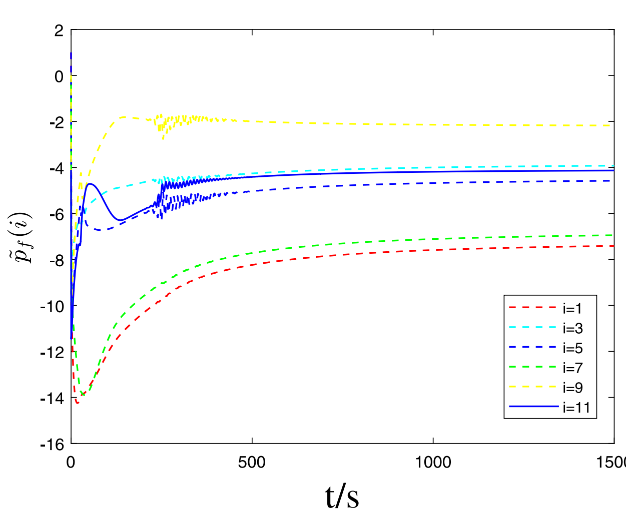


Fig. 11. Formation errors in Case 4 under $u_{aux} = 0$.

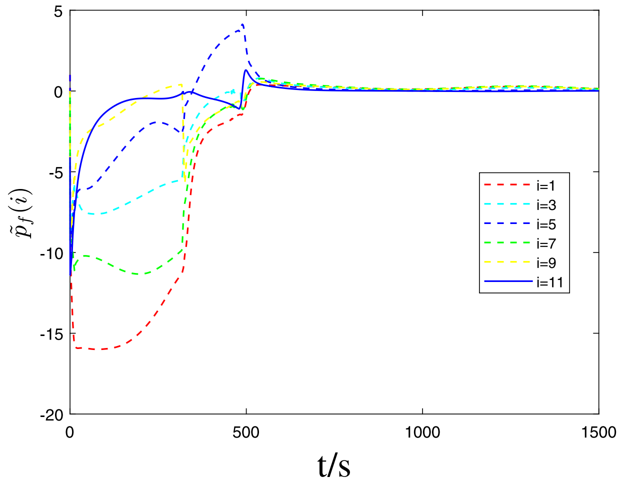


Fig. 12. Formation errors in Case 4 with $u_{aux} \neq 0$.

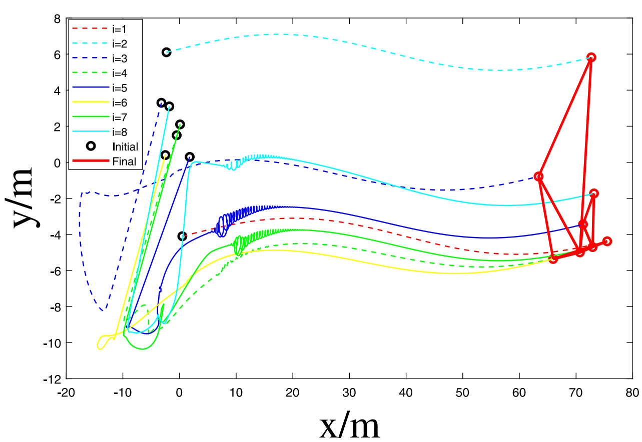


Fig. 13. Formation trajectories in Case 4 under $u_{aux} = 0$.

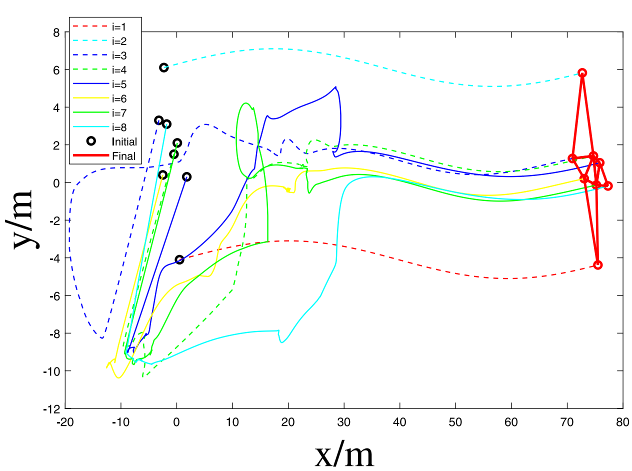


Fig. 14. Formation trajectories in Case 4 with $u_{aux} \neq 0$.

8. Conclusion and future work

This paper has proposed SLAF algorithms for 2D multi-agent systems where the followers achieve a desired formation under the condition that they only have angle measurements and communication with their neighbors. Three SLAF algorithms have been proposed when the leaders are static, moving with constant velocities, and moving with time-varying velocities, respectively. To handle the situation where some agents are collinear such that the multi-agent system becomes unlocalizable during evolution, a perturbation-based algorithm has been proposed to achieve the SLAF task with an asymptotic convergence.

To make the proposed SLAF algorithms more applicable to engineering practices, there is still a wide range of problems worthy of further investigation. We have special interests in two of them. The first is the extension of this work to 3D scenario such that the SLAF algorithms are applicable for a swarm of drones. The main challenge is to establish a new angle-induced linear equation in 3D since Eq. (2) cannot be used directly in 3D. The other is on sensor measurement noises, such as explicit analysis of the effect of stochastic noises, and the methods to mitigate the negative effect of noises.

References

Ahn, Hyo-Sung (2020). *Formation control*. Springer.

Anderson, Brian D. O. (1977). Exponential stability of linear equations arising in adaptive identification. *IEEE Transactions on Automatic Control*, *22*(1), 83–88.

Anderson, Brian D. O., Yu, Changbin, Fidan, Baris, & Hendrickx, Julien M (2008). Rigid graph control architectures for autonomous formations. *IEEE Control Systems Magazine*, *28*(6), 48–63.

Chang, Tsu-Shuan, & Chen, Chi-Tsong (1974). On the Routh-Hurwitz criterion. *IEEE Transactions on Automatic Control*, *19*(3), 250–251.

Chen, Liangming (2022). Triangular angle rigidity for distributed localization in 2D. *Automatica*.

Chen, Liangming, Cao, Ming, & Li, Chuanjiang (2021). Angle rigidity and its usage to stabilize multi-agent formations in 2D. *IEEE Transactions on Automatic Control*, *66*(8), 3667–3681.

Connelly, R., & Guest, S. D. (2015). *Frameworks, tensegrities and symmetry: understanding stable structures*. Cornell University, College of Arts and Sciences.

Diao, Yingfei, Lin, Zhiyun, & Fu, Minyue (2014). A barycentric coordinate based distributed localization algorithm for sensor networks. *IEEE Transactions on Signal Processing*, *62*(18), 4760–4771.

Fang, Xu, Li, Xiaolei, & Xie, Lihua (2021). Angle-displacement rigidity theory with application to distributed network localization. *IEEE Transactions on Automatic Control*, *66*(6), 2574–2587.

Filipov, Aleksei Fedorovich (1988). Differential equations with discontinuous right-hand side. In *Amer. math. soc* (pp. 191–231).

Fischer, Nicholas, Kamalapurkar, Rushikesh, & Dixon, Warren E. (2013). Lasalle-Yoshizawa corollaries for nonsmooth systems. *IEEE Transactions on Automatic Control*, *58*(9), 2333–2338.

Gallier, Jean, et al. (2010). The schur complement and symmetric positive semidefinite (and definite) matrices. *Penn Engineering*, 1–12.

Guo, Miao, Jayawardhana, Bayu, Lee, Jin Gyu, & Shim, Hyungbo (2020). Simultaneous distributed localization, mapping and formation control of mobile robots based on local relative measurements. *IFAC-PapersOnLine*, *53*(2), 9614–9620.

Guo, Kexin, Li, Xiuxian, & Xie, Lihua (2019). Ultra-wideband and odometry-based cooperative relative localization with application to multi-uav formation control. *IEEE Transactions on Cybernetics*, *50*(6), 2590–2603.

Han, Zhimin, Guo, Kexin, Xie, Lihua, & Lin, Zhiyun (2018). Integrated relative localization and leader–follower formation control. *IEEE Transactions on Automatic Control*, *64*(1), 20–34.

Hong, Yigwruang, Huang, Jie, & Xu, Yangsheng (2001). On an output feedback finite-time stabilization problem. *IEEE Transactions on Automatic Control*, *46*(2), 305–309.

Hu, Junyan, Niu, Hanlin, Carrasco, Joaquin, Lennox, Barry, & Arvin, Farshad (2020). Voronoi-based multi-robot autonomous exploration in unknown environments via deep reinforcement learning. *IEEE Transactions on Vehicular Technology*, *69*(12), 14413–14423.

Huang, Jiangyang, Farritor, Shane M, Qadi, Ala, & Goddard, Steve (2006). Localization and follow-the-leader control of a heterogeneous group of mobile robots. *IEEE/ASME Transactions on Mechatronics*, *11*(2), 205–215.

Jing, Gangshan, Wan, Changhuang, & Dai, Ran (2021). Angle-based sensor network localization. *IEEE Transactions on Automatic Control*.

Jing, Gangshan, Zhang, Guofeng, Lee, Heung Wing Joseph, & Wang, Long (2019). Angle-based shape determination theory of planar graphs with application to formation stabilization. *Automatica*, *105*, 117–129.

Khalil, Hassan K. (2002). *Nonlinear systems, vol. 3*. Prentice hall Upper Saddle River, NJ.

Lin, Zhiyun, Han, Zhimin, & Cao, Ming (2020). Distributed localization for multi-robot systems in presence of unlocalizable robots. In *2020 59th IEEE conference on decision and control (CDC)* (pp. 1550–1555). IEEE.

Lin, Zhiyun, Han, Tingrui, Zheng, Ronghao, & Fu, Minyue (2016). Distributed localization for 2-D sensor networks with bearing-only measurements under switching topologies. *IEEE Transactions on Signal Processing*, *64*(23), 6345–6359.

Lin, Zhiyun, Han, Tingrui, Zheng, Ronghao, & Yu, Changbin (2017). Distributed localization with mixed measurements under switching topologies. *Automatica*, *76*, 251–257.

Nguyen, Thien-Minh, Qiu, Zhirong, Nguyen, Thien Hoang, Cao, Muqing, & Xie, Lihua (2019). Persistently excited adaptive relative localization and time-varying formation of robot swarms. *IEEE Transactions on Robotics*, *36*(2), 553–560.

Oh, Kwang-Kyo, & Ahn, Hyo-Sung (2013). Formation control and network localization via orientation alignment. *IEEE Transactions on Automatic Control*, *59*(2), 540–545.

Oh, Kwang-Kyo, Park, Myoung-Chul, & Ahn, Hyo-Sung (2015). A survey of multi-agent formation control. *Automatica*, *53*, 424–440.

Ravankar, Abhijeet, Ravankar, Ankit A, Kobayashi, Yukinori, Hoshino, Yohei, Peng, Chao-Chung, & Watanabe, Michiko (2018). Hitchhiking based symbiotic multi-robot navigation in sensor networks. *Robotics*, *7*(3), 37.

Shames, Iman, Bishop, Adrian N., & Anderson, Brian D. O. (2012). Analysis of noisy bearing-only network localization. *IEEE Transactions on Automatic Control*, *58*(1), 247–252.

Shiroma, Naoji, Chiu, Yu-Huan, Sato, Noritaka, & Matsuno, Fumitoshi (2005). Cooperative task excecution of a search and rescue mission by a multi-robot team. *Advanced Robotics*, *19*(3), 311–329.

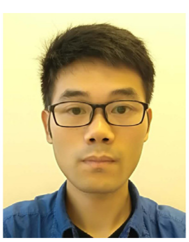
Trinh, Minh Hoang, Zelazo, Daniel, & Ahn, Hyo-Sung (2019). Pointing consensus and bearing-based solutions to the fermat–weber location problem. *IEEE Transactions on Automatic Control*, *65*(6), 2339–2354.

Ye, Mengbin, Anderson, Brian D. O., & Yu, Changbin (2017). Bearing-only measurement self-localization, velocity consensus and formation control. *IEEE Transactions on Aerospace and Electronic Systems*, *53*(2), 575–586.

Zhao, Shiyu, Li, Zhenhong, & Ding, Zhengtao (2019). Bearing-only formation tracking control of multiagent systems. *IEEE Transactions on Automatic Control*, *64*(11), 4541–4554.

Zhao, Shiyu, & Zelazo, Daniel (2016). Bearing rigidity and almost global bearing-only formation stabilization. *IEEE Transactions on Automatic Control*, *61*(5), 1255–1268.

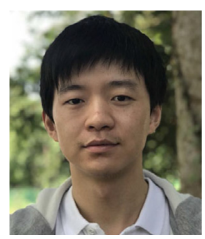
Zhao, Shiyu, & Zelazo, Daniel (2016). Localizability and distributed protocols for bearing-based network localization in arbitrary dimensions. *Automatica*, *69*, 334–341.



Liangming Chen received the B. E. degree in Automation from Southwest Jiaotong University, China, in 2015. From 2015 to 2021, he enrolled jointly in the Ph.D. program of systems and control with Harbin Institute of Technology, China and University of Groningen, the Netherlands. From 2021 to 2022, he was a Research Fellow with the School of Mechanical and Aerospace Engineering, Nanyang Technological University (NTU), Singapore. He is currently a Research Fellow with the School of Electrical and Electronic Engineering, NTU. His research interests include formation control and distributed localization.

Lihua Xie received the Ph.D. degree in electrical engineering from the University of Newcastle, Australia, in 1992. Since 1992, he has been with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, where he is currently a Professor and the Director of the Center for Advanced Robotics Technology Innovation. He served as the Head of Division of Control and Instrumentation, and the Director of Delta-NTU Corporate Lab for Cyber–Physical Systems. His research interests include robust control and estimation, networked control systems, multiagent networks, localization, and unmanned systems. He is an Editor-in-Chief for Unmanned Systems. He was the Editor of IET Book Series in Control and an Associate Editor for a number of journals including the IEEE Transactions on Automatic Control, Automatica, the IEEE Transactions on Control Systems Technology, and the IEEE Transactions on Control of Network Systems. He was an IEEE Distinguished Lecturer and an Elected Member of Board of Governors, the IEEE Control System Society. He is a Fellow of IFAC and IEEE, and Fellow of Academy of Engineering Singapore.

Xiaolei Li received the B.Eng. and Ph.D. degree in control engineering from Yanshan University, China, in 2012 and 2018, respectively. From Sep. 2018 to Mar. 2022, he acted as a Research Fellow in the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. He is currently a faculty at the School of Electrical Engineering, Yanshan University, Qinhuangdao, China. His current research interests are focused on localization and formation control of robotic systems and cyber security of cyber-physical systems.



Xu Fang received the B.Eng. degree in electrical engineering from Beijing Forestry University, Beijing, China, in 2013, the M.Sc. degree in control theory and control engineering from Beihang University, Beijing, in 2016, and the Ph.D. degree in the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore in 2021. Currently, he is a research fellow in Nanyang Technological University, Singapore. His research interests include cooperative control and network localization.

Mir Feroskhan received first-class honors in aerospace engineering from Nanyang Technological University, Singapore, in 2011, and the Ph.D. degree in aerospace engineering from the Florida Institute of Technology, Melbourne, FL, in 2016. He is currently an assistant professor with the School of Mechanical and Aerospace Engineering at NTU. His research interests include nonlinear control systems, VTOL systems, drone design and development, flight dynamics and control, and aerial robotics.