This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2022.3151567, IEEE Transactions on Automatic Control

Globally Stabilizing Triangularly Angle Rigid Formations

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***Abstract*—The global stabilization of planar angle rigid formations is acknowledged to be a challenging problem in the existing literature even when relative position measurements are available among neighboring agents. Inspired by an angle-induced linear constraint existing in each triangle, this paper proposes formation control laws to achieve global stabilization of triangularly angle rigid formations using local relative position measurements. Compared to some other globally stable formation control systems using local relative position measurements, our approach is shown to be more computationally effective and scalable. Moreover, by additionally controlling the relative position between a pair of neighboring agents, we propose modified formation control laws to globally stabilize triangularly angle rigid formation with prescribed orientation and scale. Compared to other formation stabilization approaches with prescribed orientation and scale, the proposed formation control law guarantees global stability instead of almost global stability. Finally, we remark that the proposed approach can also be used to globally stabilize triangular formations specified by ratio-of-distance constraints. Simulation examples validate the effectiveness of the proposed formation control approaches.**

***Index Terms*—Angle rigidity, triangular formations, formation control, global stabilization, multi-agent system.**

I. Introduction

The topic of multi-agent formations has been extensively studied due to its wide applications in practical missions. The aim of multi-agent formations is to achieve a desired geometric shape for a group of agents by using available sensing and communication information [1]. Two aspects have been mainly concerned for the study of multi-agent formations, i.e., controller design using available information and convergence property of the formation control system with the designed controllers [1].

For the first aspect, the measurements used for multi-agent formations mainly include absolute positions, relative positions, bearings, distances, ratio of distances, and angles. These measurements can be accessed via sensors, such as GPS, compass, radar, camera, Ultra Wideband, and sensor array, etc [1]. When absolute positions or aligned relative positions are available, a desired formation can be achieved based on the linear property of these measurements with respect to agents' positions [2]. However, absolute positions obtained from GPS module are unavailable in indoor environment [1] and the aligned relative positions require all agents' coordinate frames to have the same orientation where an undesired translational velocity and distorted formation will show up when a small orientation misalignment in agents' coordinate frames exists [3]. To avoid these deficiencies, inter-agent measurements that are independent of the global coordinate frame and the orientations of the agents' coordinate frames are more favorable, which include local relative positions [4]–[7], distances [8], and angles [9], [10]. Among these measurements, local relative position measurements have been widely used to achieve a desired formation specified by distances [11], ratio of distances [12], or angles [13]. Therefore, it is important to stabilize an angle rigid formation by also using local relative position measurements [13].

For the second aspect, the convergence property of multi-agent formations includes local convergence and global convergence, which correspond to different attraction regions that determine initial formation errors for stability guarantee. For linear controllers using the measurements of absolute positions or aligned relative positions, the global convergence is guaranteed when the closed-loop dynamics of the formation control system is asymptotically stable. However, for nonlinear controllers using the measurements of local relative positions [11], [14], [15], aligned bearings [16], distances [8], and angles [9], many of the designed formation controllers only guarantee local or almost global stability. In particular, when a formation is specified by angle constrains [9], [13], [17], [18], the global stabilization of angle rigid formations has not been achieved yet. Note that multi-agent formations with local stability require the initial formations to be sufficiently close to the desired formation, which limit their applications in practice. Since multi-agent formations with global stability are much more preferable [14], [15], it is also important to design distributed formation controllers to globally stabilize angle rigid formations.

Motivated by the above-mentioned aspects, this paper aims to globally stabilize angle rigid formations using local relative position measurements. Inspired by a transformation from geometric constraints among agents into linear algebraic constraints, we show that angle-induced linear constraints existing in triangles can be efficiently used for the global stabilization of triangularly angle rigid formations. Firstly, we focus on the global stabilization of triangularly angle rigid formations using local relative position measurements. Then, we consider the other case that by additionally controlling the relative position between a pair of neighboring agents, a triangularly angle rigid formation can be globally stabilized with a prescribed orientation and scale. Compared to the existing results on stabilizing angle rigid formations with local convergence [9], [13], [17], [18] or almost global convergence [10], [19], our proposed formation control laws can globally stabilize triangularly angle

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rigid formations where the prescribed orientation and scale can also be achieved.

The rest of the paper is organized as follows. Section II presents the preliminaries. Section III discusses global stabilization of angle rigid formations using local relative position measurements. Section IV studies the global stabilization of angle rigid formations with the prescribed orientation and scale. Section V discusses the extension to the global stabilization of formations specified by ratio-of-distance constrains. Simulations are provided in Section VI.

II. Preliminaries

A. Notations

Consider a planar multi-agent system consisting of $n \geq 3$ agents. Let $\mathcal{V} = \{1, 2, ..., n\}$ be the set of the agents which are labeled from 1 to n. Denote agent i's position by $p_i \in \mathbb{R}^2, i \in \mathcal{V}$ and let $p = [p_1^\top, p_2^\top, ..., p_n^\top]^\top \in \mathbb{R}^{2n}$. Let I_2, 1_n, and $\otimes$ be the 2-by-2 identity matrix, $n \times 1$ column vector of all ones, the Kronecker product, respectively. Denote $R(\theta) := \left[\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right]$ as the 2D rotation matrix with rotation angle $\theta \in \mathbb{R}$. In this paper, we assume that each agent holds an unknown but fixed coordinate frame $\sum_i$ to conduct the relative position measurements with respect to its neighbors. Define $\sum_g$ as the global coordinate frame and let $R_g^i \in SO(2)$ be the rotation matrix describing the rotation from $\sum_g$ to $\sum_i$.

B. Triangularly angle rigid formations

As introduced in [9], each angle constraint is associated with three vertices whose description by a graph is inconvenient. Instead, we use the notion of *angularity* to describe the multi-agent formations with angle constraints. For the vertex set $\mathcal{V} = \{1, 2, ..., n\}$ where node $i \in \mathcal{V}$ corresponds to agent i, we define a three-vertex *triplet* (i, j, k) to describe the angle constraint $\angle ijk$, which is equivalent to constraining $\angle kji$, and thus (i, j, k) and (k, j, i) can be interchangeable with each other. Then, we define $\mathcal{A} \subset \mathcal{V} \times \mathcal{V} \times \mathcal{V} = \{(i, j, k), i, j, k \in \mathcal{V}, i \neq j \neq k\}$ as an angle set, each element of which is a triplet. The combination of the vertex set $\mathcal{V}$, the angle set $\mathcal{A}$ and the position configuration $p \in \mathbb{R}^{2n}$ is called an *angularity* which we denote by $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ [9]. We say $\mathcal{A}$ is a *triangular angle set* if for every $(i_1, j_1, k_1) \in \mathcal{A}$, there also exists either $(j_1, k_1, i_1) \in \mathcal{A}$ or $(k_1, i_1, j_1) \in \mathcal{A}$. We say $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is a *triangular angularity* if $\mathcal{A}$ is a triangular angle set. The number of triangles in a triangular angularity $\mathbb{A}$ is denoted by $m \in \mathbb{N}^+$. If $(i, j, k) \in \mathcal{A}$, then $\{j, k\} \in \mathcal{N}_i, \{i, k\} \in \mathcal{N}_j, \{i, j\} \in \mathcal{N}_k$ where $\mathcal{N}_i$ represents i's neighbor set.

For three non-coincident agents k, i, j, we define the signed interior angle $\alpha_{kij} \in [0, 2\pi)$ among agents k, i, j as [9]

$$\alpha_{kij} := \begin{cases} \arccos(b_{ij}^\top b_{ik}), & \text{if } b_{ij}^\top b_{ik}^\perp \geq 0, \\ 2\pi - \arccos(b_{ij}^\top b_{ik}), & \text{otherwise}, \end{cases} \tag{1}$$

where $b_{ij} := (p_j - p_i)/\|p_j - p_i\|$ is the bearing from agent i to agent j which is a unit vector, $i, j \in \mathcal{V}$, $b_{ik}^\perp = R(\frac{\pi}{2}) b_{ik} = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] b_{ik}$. Note that α_{kij} represents the angle rotating from the ray $\overrightarrow{ik}$ to the ray $\overrightarrow{ij}$ under the counterclockwise direction. Following [9], an angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is said to be angle rigid if under a small perturbation of p, the magnitude of all the angles defined in $\mathcal{A}$ keeps the same. Then, we say an angle-constrained multi-agent formation is angle rigid if its corresponding angularity is angle rigid. In this paper, we are interested in the stabilization of triangularly angle rigid formation which consists of $2m$ independent angle constraints derived from m triangles among the n agents. Therefore, the desired angle rigid formation in this paper can be described by a set of angle constraints

$$f_{\mathcal{A}}(\alpha^*) := [\cdots, \alpha_{kij}^*, \cdots]^\top \in \mathbb{R}^{2m}, \ (k, i, j) \in \mathcal{A} \tag{2}$$

where $\alpha_{kij}^* \in [0, 2\pi)$.

C. Angle-induced linear constraints in triangles

To make preparations for the control design, we introduce how to transfer the nonlinear angle constraint into linear algebraic equation by introducing an angle-induced linear constraint existing in each triangle.

By taking three angle constraints $\alpha_{kij}, \alpha_{ijk}, \alpha_{jki}$ from the non-degenerate triangle $\triangle ijk$ as an example [20], one has

$$(p_j - p_i)/\|p_j - p_i\| = R(\alpha_{kij})(p_k - p_i)/\|p_k - p_i\|. \tag{3}$$

Using the law of sines $\frac{\|p_k - p_i\|}{\|p_j - p_i\|} = \frac{\sin \alpha_{ijk}}{\sin \alpha_{jki}}$ and the fact (3), the angle-induced linear constraint in $\triangle ijk$ can be written as [20]

$$\begin{aligned} f_i^{\triangle ijk}(\alpha, p) &= A_i^{\triangle ijk}(\alpha) p_i + A_j^{\triangle ijk}(\alpha) p_j + A_k^{\triangle ijk}(\alpha) p_k \\ &= \sin \alpha_{jki}(p_i - p_k) - \sin \alpha_{ijk} R^\top(\alpha_{kij})(p_i - p_j) = 0 \end{aligned} \tag{4}$$

where the coefficient matrices

$$\begin{aligned} & A_i^{\triangle ijk}(\alpha) = \left(\sin \alpha_{jki} I_2 - \sin \alpha_{ijk} R^\top(\alpha_{kij})\right) \in \mathbb{R}^{2\times 2}, \\ & A_j^{\triangle ijk}(\alpha) = \sin \alpha_{ijk} R^\top(\alpha_{kij}) \in \mathbb{R}^{2\times 2}, \\ & A_k^{\triangle ijk}(\alpha) = -\sin \alpha_{jki} I_2 \in \mathbb{R}^{2\times 2} \end{aligned}$$

are only related with the interior angles $\alpha_{jki}, \alpha_{ijk}, \alpha_{kij}$. Note that for a degenerate triangle $\triangle ijk$, i.e., p_i, p_j, p_k are collinear, the angle-induced linear constraint (4) degrades into a trivial equation. Thus, to avoid the collinearity among agents, the configuration p is assumed to be generic, which follows the definition in [21, Section 1.2].

For a triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with multiple triangles defined in $\mathcal{A}$, one can write all the angle-induced linear constraints (4), i.e., $\forall (i, j, k) \in \mathcal{A}$, from the triangular angularity into a compact form

$$R_{\mathcal{A}}(\alpha) p = 0 \tag{5}$$

where $R_{\mathcal{A}}(\alpha) \in \mathbb{R}^{2m \times 2n}$ can be written in the form of [20]

$$\begin{array}{c} \\ \text{1st } \triangle \\ \cdots \\ \triangle ijk \\ \cdots \\ m\text{th } \triangle \end{array} \begin{array}{c} \begin{array}{ccccccc} \cdots & \text{Vertex } i & \cdots & \text{Vertex } j & \cdots & \text{Vertex } k & \cdots \end{array} \\ \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & A_i^{\triangle ijk} & 0 & A_j^{\triangle ijk} & 0 & A_k^{\triangle ijk} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array} \tag{6}$$

whose rows are indexed by the triangles defined in the triangular angle set $\mathcal{A}$ and columns are indexed by the vertices in $\mathcal{V}$. According to [9], [22], the maximum rank of $R_{\mathcal{A}}(\alpha)$ is $2n-4$. Now, we have the following lemma.

Lemma 1. *For a triangular angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p)$ with generic p, if $\mathbb{A}$ is angle rigid, then* $\text{Rank}(R_{\mathcal{A}}(\alpha(p))) = 2n-4$ *and* $\text{Null}(R_{\mathcal{A}}(\alpha(p))) = \text{Span}\{p, (I_n \otimes R(\frac{\pi}{2}))p, 1_n \otimes [1,0]^\top, 1_n \otimes [0,1]^\top\}$.

Proof. Suppose on the contrary that $\text{Rank}(R_{\mathcal{A}}(\alpha(p))) \neq 2n-4$. Since $\text{Rank}(R_{\mathcal{A}}(\alpha(p))) \leq 2n-4$, one has $\text{Rank}(R_{\mathcal{A}}(\alpha(p))) < 2n-4$, which implies that the null space of $R_{\mathcal{A}}(\alpha(p))$ contains other configurations besides the scaling, rotation, translation along the X-axis, and translation along the Y-axis with respect to p [20]. Since p is generic, according to the definition of angle rigidity, one has that $\mathbb{A}$ is not angle rigid, which implies a contradiction with the assumption. □

Remark 1. *Since the minimum number of angle constraints to guarantee angle rigidity of a triangular angularity with generic configuration is $2n-4$ [9], [13], at least $m=n-2$ triangles are needed and the rank condition developed in [9] can be used to construct a triangularly angle rigid formation.*

III. Global stabilization of angle rigid formations

In this paper, we assume that the agents are governed by single-integrator dynamics

$$\dot{p}_i(t) = u_i(t), i=1,...,n \tag{7}$$

where $p_i \in \mathbb{R}^2$ represents agent i's absolute position in $\sum_g$, and $u_i \in \mathbb{R}^2$ is the control input to be designed. Consider that the desired formation among agents is specified by the angle constraints in $f_{\mathcal{A}}(\alpha^*)$ whose corresponding desired angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p^*)$ is angle rigid, where $p^* \in \mathbb{R}^{2n}$ is one of the generic formation configurations that satisfies all the angle constraints defined in $f_{\mathcal{A}}(\alpha^*)$. Given the angle function $f_{\mathcal{A}}(\alpha^*)$, the aim of angle rigid formation control is to design u_i for (7) such that

$$\lim_{t\to\infty}(\alpha_{ijk}(t) - \alpha^*_{ijk}) = 0, \forall (i,j,k) \in \mathcal{A}. \tag{8}$$

In the following, we first design the formation control law, then analyze its properties.

A. Formation controller design and stability analysis

According to Lemma 1, if the desired triangular angularity $\mathbb{A}(\mathcal{V},\mathcal{A},p^*)$ is angle rigid, then the kernel of $R_{\mathcal{A}}(\alpha^*)$ is spanned by $\{p^*, (I_n \otimes R(\frac{\pi}{2}))p^*, 1_n \otimes [1,0]^\top, 1_n \otimes [0,1]^\top\}$. Now, we define a new matrix

$$\mathcal{L}(\alpha) = R^\top_{\mathcal{A}}(\alpha) R_{\mathcal{A}}(\alpha) \in \mathbb{R}^{2n\times 2n}. \tag{9}$$

Then, one has the following results.

Lemma 2. *If $\mathbb{A}^*(\mathcal{V},\mathcal{A},p^*)$ is angle rigid, then $\mathcal{L}(\alpha^*)$ is positive semi-definite and has four zero eigenvalues whose corresponding eigenvectors are $\{p^*, (I_n \otimes R(\frac{\pi}{2}))p^*, 1_n \otimes [1,0]^\top, 1_n \otimes [0,1]^\top\}$.*

Proof. Since $\mathcal{L}(\alpha^*) = R^\top_{\mathcal{A}}(\alpha^*) R_{\mathcal{A}}(\alpha^*)$, $\mathcal{L}(\alpha^*)$ is positive semi-definite. The kernel of $R_{\mathcal{A}}(\alpha^*)$ is the same as the kernel of $\mathcal{L}(\alpha^*)$, thus is spanned by the four linearly independent vectors $\{p^*, R(\frac{\pi}{2})p^*, 1_n \otimes [1,0]^\top, 1_n \otimes [0,1]^\top\}$, which correspond to the scaling, rotation, translation along the X-axis, and translation along the Y-axis with respect to p^*, respectively. □

Based on the constant matrix $\mathcal{L}(\alpha^*)$, we design the linear formation control law for (7) in the compact form as

$$u(t) = -\mathcal{L}(\alpha^*)p(t) \tag{10}$$

where $u(t) = [u_1^\top, ..., u_n^\top]^\top \in \mathbb{R}^{2n}$, and $p(t) = [p_1^\top, ..., p_n^\top]^\top \in \mathbb{R}^{2n}$. According to (6) and (9), the component form of the formation controller (10) can be written as

$$\begin{aligned} u_i(t) = &-\Big[\sum\nolimits_{(i,j_1,k_1)\in\bar{\mathcal{A}}} (A_i^{\triangle ij_1k_1}(\alpha^*))^\top f_i^{\triangle ij_1k_1}(\alpha^*, p(t)) \\ &+ \sum\nolimits_{(j_2,i,k_2)\in\bar{\mathcal{A}}} (A_i^{\triangle j_2ik_2}(\alpha^*))^\top f_i^{\triangle j_2ik_2}(\alpha^*, p(t)) \\ &+ \sum\nolimits_{(j_3,k_3,i)\in\bar{\mathcal{A}}} (A_i^{\triangle j_3k_3i}(\alpha^*))^\top f_i^{\triangle j_3k_3i}(\alpha^*, p(t))\Big], \end{aligned} \tag{11}$$

where $f_i^{\triangle ij_1k_1}(\alpha^*, p(t)) = \sin\alpha^*_{j_1k_1i}(p_i(t) - p_{k_1}(t)) - \sin\alpha^*_{ij_1k_1} R^\top(\alpha^*_{k_1ij_1})(p_i(t) - p_{j_1}(t))$ is the weighted sum of the relative position measurements $(p_i - p_{k_1})$ and $(p_i - p_{j_1})$, $\{k_1, j_1\} \in \mathcal{N}_i$, and $f_i^{\triangle j_2ik_2}(\alpha^*, p(t)) = \sin\alpha^*_{ik_2j_2}(p_{j_2}(t) - p_{k_2}(t)) - \sin\alpha^*_{j_2ik_2} R^\top(\alpha^*_{k_2j_2i})(p_{j_2}(t) - p_i(t))$, and $f_i^{\triangle j_3k_3i}(\alpha^*, p(t)) = \sin\alpha^*_{k_3ij_3}(p_{j_3}(t) - p_i(t)) - \sin\alpha^*_{j_3k_3i} R^\top(\alpha^*_{ij_3k_3})(p_{j_3}(t) - p_{k_3}(t))$, $\bar{\mathcal{A}} \subset \mathcal{A}, |\bar{\mathcal{A}}| = m$, and if $(i,j,k) \in \bar{\mathcal{A}}$, then $(j,k,i) \notin \bar{\mathcal{A}}, (k,i,j) \notin \bar{\mathcal{A}}$. Since the desired angles are known at the control design stage, only relative position measurements are needed and no communication is needed for each agent to implement the control (11). The non-adjacent relative position vector $p_{j_2}(t) - p_{k_2}(t) = p_{j_2}(t) - p_i(t) + p_i(t) - p_{k_2}(t)$ can be obtained by agent i's measurements of $p_{j_2}(t) - p_i(t)$ and $p_i(t) - p_{k_2}(t)$. Moreover, from (9), agents have an undirected measurement topology under (10).

Theorem 1. *If the desired triangular angularity $\mathbb{A}^*(\mathcal{V},\mathcal{A},p^*)$ is angle rigid, then under the control law* (11)*, $p(t)$ will globally converge to $\beta_1 p^* + \beta_2(I_n \otimes R(\frac{\pi}{2}))p^* + \beta_3 1_n \otimes [1,0]^\top + \beta_4 1_n \otimes [0,1]^\top$, where $\beta_i \in \mathbb{R}, i=1,2,3,4$, and for almost all initial conditions $p(0)$,* (8) *is almost globally achieved.*

Proof. Substituting the control law (10) into (7), one has

$$\dot{p}(t) = -\mathcal{L}(\alpha^*)p(t) \tag{12}$$

Since (12) is a linear system and $\mathcal{L}(\alpha^*)$ is positive semi-definite, one has that $p(t)$ will globally and asymptotically converge to the null space of $\mathcal{L}(\alpha^*)$, which is the linear combination of $\{p^*, R(\frac{\pi}{2})p^*, 1_n \otimes [1,0], 1_n \otimes [0,1]\}$, i.e.,

$$p(t) \to \beta_1 p^* + \beta_2(I_n \otimes R(\frac{\pi}{2}))p^* + \beta_3 1_n \otimes \begin{bmatrix}1\\0\end{bmatrix} + \beta_4 1_n \otimes \begin{bmatrix}0\\1\end{bmatrix}$$

Since $\mathcal{L}(\alpha^*)$ is positive semi-definite, there exists a nonsingular matrix $Q \in \mathbb{R}^{2n\times 2n}$ with its first four columns being $p^*, (I_n \otimes R(\frac{\pi}{2}))p^*, 1_n \otimes [1,0]^\top, 1_n \otimes [0,1]^\top$ such that

$$Q^{-1}\mathcal{L}(\alpha^*)Q = \begin{bmatrix} 0_{4\times 4} & 0 \\ 0 & \Delta \end{bmatrix} \tag{13}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2022.3151567, IEEE Transactions on Automatic Control

where $\Delta \in \mathbb{R}^{(2n-4)\times(2n-4)}$ is positive definite. By defining a coordinate transformation $z(t) = Q^{-1}p(t) = [z_1, z_2, z_3, z_4, \bar{z}^\top]^\top$ with $z_i \in \mathbb{R}, i = 1, ..., 4$ and $\bar{z} \in \mathbb{R}^{2n-4}$, the dynamics (12) can be written as

$$\dot{Y} = \begin{bmatrix}\dot{z}_1 & \dot{z}_2 & \dot{z}_3 & \dot{z}_4 & \dot{\bar{z}}\end{bmatrix}^\top = -\begin{bmatrix}0_{4\times 4} & 0 \\ 0 & \Delta\end{bmatrix} Y \tag{14}$$

which implies $z_i(t) = z_i(0) = \beta_i, i = 1, ..., 4$ and $\bar{z}(t)$ globally and exponentially fast converges to 0 as $t \to \infty$. It follows that $p(t)$ globally and exponentially fast converges to $z_1(0)p^* + z_2(0)(I_n \otimes R(\frac{\pi}{2}))p^* + z_3(0)1_n \otimes [1, 0]^\top + z_4(0)1_n \otimes [0, 1]^\top$. Suppose that $q_1^\top, q_2^\top$ are the first two rows of Q^{-1}. Then, according to the coordinate transformation $z(t) = Q^{-1}p(t)$, $z_1(0) \neq 0$ if $q_1^\top p(0) \neq 0$, and $z_2(0) \neq 0$ if $q_2^\top p(0) \neq 0$. Since q_1, q_2 are linearly independent, $q_1^\top p(0) = 0$ and $q_2^\top p(0) = 0$ could not occur at the same time for almost all initial conditions $p(0)$. In other words, since $q_1^\top p(0) \neq 0$ or $q_2^\top p(0) \neq 0$ holds for almost all initial positions, at least one of β_1, β_2 is nonzero, i.e., the desired triangularly angle rigid formation is almost globally achieved. □

Remark 2. *Compared to [9], [13], [17]–[19] where relative position or direction measurements are used to stabilize angle rigid formations with local convergence, the designed control law* (10) *can guarantee global convergence. Compared to sequential formations [9], [19], the desired formations described by* (2) *only need to be triangularly angle rigid. Based on complex Laplacian, a desired formation is globally achieved in [7] if agents' initial positions are not orthogonal to one special vector. Compared to [7], the desired formation is globally achieved under* (10) *if agents' initial positions are not orthogonal to two special vectors at the same time.*

B. Other properties of the proposed formation control law

In this subsection, we further analyze three properties of the proposed formation control law (10).

1) Robustness against coordinate frames' misalignment: Consider that all agents' coordinate frames have different orientations. Then, $f_i^{\triangle ij_1k_1}(\alpha^*, p)$ measured in agent i's local coordinate frame $\sum_i$ becomes $R_g^i f_i^{\triangle ij_1k_1}(\alpha^*, p)$. Because $A_i^{\triangle ij_1k_1}(\alpha^*)R_g^i = R_g^i A_i^{\triangle ij_1k_1}(\alpha^*)$, $R^\top(\alpha^*_{k_1ij_1})R_g^i(p_i - p_{j_1}) = R_g^i R^\top(\alpha^*_{k_1ij_1})(p_i - p_{j_1})$, it can be easily verified by following [9] that the control law (11) can be implemented in each agent's local coordinate frame. That is to say, to implement (11), each agent can have its own local coordinate frame $\sum_i$ to obtain the relative position measurements with respect to its neighbors.

2) Computational effectiveness: The matrix $\mathcal{L}(\alpha^*)$ defined in (9) is only related to the desired angles α^* of the formation instead of the desired formation configuration p^* as required in [7], [23], [24]. When p^* is given, the calculation of α^* from p^* is straightforward by following (1). However, when α^* is given, the calculation of p^* from α^* is NP-hard [25] or needs extra computations and iterations, e.g., by employing a network localization law to produce a p^*. Therefore, our approach is computationally effective when either p^* or α^* is available, while the approaches in [7], [23], [24] are computationally effective only when p^* is available.

3) Scalability: The operations of removing agents from the formation and adding agents into the formation depend on the scalability of the proposed formation control law. In [7], [23], [24], the control gain matrices or a stress matrix in the formation control laws are designed using the information of p^*, which implies that when conducting the operations of removing and adding agents, the gain matrices need to be redesigned and recomputed in a centralized manner. Different from [7], [23], [24], the control gain matrix in our formation control law (10) only depends on the desired angles with respect to agents' neighbors. Therefore, when conducting the operations of removing or adding agents, only associated agents need to delete the related control components or add the related control components in the control law, respectively. Take a 4-agent angle-constrained formation as an example with agents labeled by 1,2,3,4, and suppose that $\mathcal{A}$ contains two triangles $\triangle 123$ and $\triangle 234$. Then, following (11), the control law for agent 3 is

$$\begin{aligned} u_3(t) =& -(A_3^{\triangle 123}(\alpha^*))^\top f_3^{\triangle 123}(\alpha^*, p(t)) \\ & -(A_3^{\triangle 234}(\alpha^*))^\top f_3^{\triangle 234}(\alpha^*, p(t)) \end{aligned} \tag{15}$$

When agent 4 is removed from the formation, agent 3 only needs to remove the control component $-(A_3^{\triangle 234}(\alpha^*))^\top f_3^{\triangle 234}(\alpha^*, p(t))$ from (15), and agent 1 does not need to change anything. It can be checked that the formation after this remove is still globally stable in the sense that all the desired angles in $\triangle 123$ will be achieved.

IV. Global stabilization of angle rigid formations with prescribed orientation and scale

This section aims to globally stabilize angle rigid formations with prescribed orientation and scale by controlling some neighboring agents' relative positions. Without loss of generality, we assume that agents 1 and 2 are neighboring agents, and the formation's orientation and scale are represented by the relative position between agents 1 and 2. In the following, we firstly consider that agents 1 and 2 are assigned to control their relative position, and then the case that only agent 1 is assigned to control its relative position with respect to agent 2. Let nonzero $\delta_{12}^* = -\delta_{21}^* \in \mathbb{R}^2$ be the desired relative position of agent 2 with respect to agent 1 under which the formation's desired scale is $\|\delta_{12}^*\| \neq 0$ and desired orientation is $\delta_{12}^*/\|\delta_{12}^*\|$.

A. Both agents 1 and 2 control the inter-agent relative position

We propose the following control laws for all the agents

$$\begin{aligned} u_i(t) = -[&\textstyle\sum_{(i,j_1,k_1)\in\bar{\mathcal{A}}}(A_i^{\triangle ij_1k_1}(\alpha^*))^\top f_i^{\triangle ij_1k_1}(\alpha^*, p(t)) \\ &+ \textstyle\sum_{(j_2,i,k_2)\in\bar{\mathcal{A}}}(A_i^{\triangle j_2ik_2}(\alpha^*))^\top f_i^{\triangle j_2ik_2}(\alpha^*, p(t)) \\ &+ \textstyle\sum_{(j_3,k_3,i)\in\bar{\mathcal{A}}}(A_i^{\triangle j_3k_3i}(\alpha^*))^\top f_i^{\triangle j_3k_3i}(\alpha^*, p(t))] - b_i(t) \end{aligned} \tag{16}$$

where $k_1 > 0$ is an *arbitrary positive* gain, and $b_i(t) = \begin{cases} k_1(p_1(t) - p_2(t) - \delta_{21}^*), & \text{if } i = 1 \\ k_1(p_2(t) - p_1(t) - \delta_{12}^*), & \text{if } i = 2 \\ 0, & \text{otherwise} \end{cases}$. Following (11), the closed-loop dynamics under (16) can be written as

$$\dot{p}(t) = -\mathcal{L}(\alpha^*)p(t) - B(p(t) - \delta^*) \tag{17}$$

where $\delta^* = [\delta_1^{*\top}, ..., \delta_n^{*\top}]^\top \in \mathbb{R}^{2n}$ is one of the desired formation configurations satisfying $\delta_1^* - \delta_2^* = \delta_{21}^*, \delta_2^* - \delta_1^* = \delta_{12}^*, \alpha(\delta^*) = \alpha^*$, and

$$B = \begin{bmatrix} k_1 I_2 & -k_1 I_2 & 0 & \cdots & 0 \\ -k_1 I_2 & k_1 I_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \tag{18}$$

Note that the configuration vector δ^* is not required for the design of the formation control law (16), but only for the stability analysis. Since $\delta^* \subseteq \text{Null}(R_{\mathcal{A}}(\alpha^*))$, one has $\delta^* \subseteq \text{Null}(\mathcal{L}(\alpha^*))$. Thus, (17) can be rewritten as

$$\dot{p}(t) = -\left(\mathcal{L}(\alpha^*) + B\right)\left(p(t) - \delta^*\right) \tag{19}$$

We split all agents into two groups, namely agents 1, 2 (termed leader agent group) and agents 3 to n (termed follower agent group). Then, the matrix $R_{\mathcal{A}}(\alpha^*)$ can be partitioned as $R_{\mathcal{A}}(\alpha^*) = [R^l_{\mathcal{A}}(\alpha^*), R^f_{\mathcal{A}}(\alpha^*)]$ where $R^l_{\mathcal{A}}(\alpha^*) \in \mathbb{R}^{2m\times 4}$ and $R^f_{\mathcal{A}}(\alpha^*) \in \mathbb{R}^{2m\times(2n-4)}$. Correspondingly, the matrix $\mathcal{L}(\alpha^*)$ can be partitioned into

$$\mathcal{L}(\alpha^*) = \begin{bmatrix} \mathcal{L}_{ll} & \mathcal{L}_{lf} \\ \mathcal{L}_{fl} & \mathcal{L}_{ff} \end{bmatrix} \tag{20}$$

where $\mathcal{L}_{ll} = (R^l_{\mathcal{A}})^\top R^l_{\mathcal{A}}, \mathcal{L}_{lf} = (R^l_{\mathcal{A}})^\top R^f_{\mathcal{A}}, \mathcal{L}_{fl} = (R^f_{\mathcal{A}})^\top R^l_{\mathcal{A}}$, and $\mathcal{L}_{ff} = (R^f_{\mathcal{A}})^\top R^f_{\mathcal{A}}$.

Lemma 3. *For a triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with generic p, if $\text{Rank}(R_{\mathcal{A}}(\alpha(p))) = 2n-4$, then $\text{Rank}(R^f_{\mathcal{A}}(\alpha(p))) = 2n-4$.*

Proof. According to (5), one has $R_{\mathcal{A}}(\alpha(p))p = 0$ which is the compact form of all angle-induced linear constraints in a triangular angularity. It follows from [20], [22], [26] that if $\text{Rank}(R_{\mathcal{A}}(\alpha(p))) = 2n-4$, then the triangular angularity is localizable, i.e., when arbitrary two nodes in $\mathcal{V}$ are selected as anchor nodes, the remaining nodes' positions can be uniquely determined. More specifically, if $p_a = [p_1^\top, p_2^\top]^\top$ is given, then $p_f = [p_3^\top, ..., p_n^\top]^\top$ has a unique solution under the fact (5). Note that (5) can be rewritten as [20]

$$[R^l_{\mathcal{A}}(\alpha), R^f_{\mathcal{A}}(\alpha)] \begin{bmatrix} p_l \\ p_f \end{bmatrix} = R^l_{\mathcal{A}}(\alpha)p_l + R^f_{\mathcal{A}}(\alpha)p_f = 0 \tag{21}$$

For the linear equation $R^f_{\mathcal{A}}(\alpha)p_f = -R^l_{\mathcal{A}}(\alpha)p_l$ from (21), when p_l is given, p_f has a unique solution, which implies that $\text{Rank}(R^f_{\mathcal{A}}(\alpha(p))) = 2n-4$. □

Now, we present the main results.

Theorem 2. *If the desired triangular angularity $\mathbb{A}^*(\mathcal{V}, \mathcal{A}, p^*)$ is angle rigid, then under the control law (16), $p(t)$ will globally converge to $\delta^* + \beta_5 1_n \otimes [1,0]^\top + \beta_6 1_n \otimes [0,1]^\top, \beta_5 \in \mathbb{R}, \beta_6 \in \mathbb{R}$, i.e., the desired angle rigid formation is almost globally achieved with the prescribed orientation and scale.*

Proof. Note that the closed-loop dynamics (19) is linear. Since $\mathcal{L}(\alpha^*)$ and B are positive semi-definite, $\mathcal{L}(\alpha^*) + B$ is also positive semi-definite. Therefore, to evaluate the stability of (19), we only need to check the null space of $\mathcal{L}(\alpha^*) + B$. Since the sum of each row of $\mathcal{L}(\alpha^*) + B$ is zero, the two vectors $1_n \otimes [1,0]^\top$ and $1_n \otimes [0,1]^\top$ are in the null space of $\mathcal{L}(\alpha^*) + B$. Now, we prove that there is no other nonzero vectors lying in the null space of $\mathcal{L}(\alpha^*) + B$.

Since $\mathbb{A}^*$ is angle rigid, one has $\text{Rank}(R_{\mathcal{A}}(\alpha^*)) = 2n-4$ from Lemma 1. It follows from Lemma 3 that $\text{Rank}(R^f_{\mathcal{A}}(\alpha^*)) = 2n-4$. Then, $\text{Rank}(\mathcal{L}_{ff}) = 2n-4$ according to (20). It follows that $\text{Rank}([\mathcal{L}_{fl}, \mathcal{L}_{ff}]) = 2n-4$ because $[\mathcal{L}_{fl}, \mathcal{L}_{ff}]$ is a sub-matrix of $\mathcal{L}(\alpha^*)$ and $\text{Rank}(\mathcal{L}(\alpha^*)) \le 2n-4$. Then, the null space of $[\mathcal{L}_{fl}, \mathcal{L}_{ff}]$ is spanned by $\{p^*, (I_n \otimes R(\frac{\pi}{2}))p^*, 1_n \otimes [1,0], 1_n \otimes [0,1]\}$.

Note that $\mathcal{L}(\alpha^*) + B = \begin{bmatrix} \mathcal{L}_{ll}+\bar{B} & \mathcal{L}_{lf} \\ \mathcal{L}_{fl} & \mathcal{L}_{ff} \end{bmatrix}$, $\bar{B} = \begin{bmatrix} k_1 I_2 & -k_1 I_2 \\ -k_1 I_2 & k_1 I_2 \end{bmatrix}$ which implies that $[\mathcal{L}_{fl}, \mathcal{L}_{ff}]$ is also a sub-matrix of $\mathcal{L}(\alpha^*) + B$. Therefore, only $\{p^*, (I_n \otimes R(\frac{\pi}{2}))p^*, 1_n \otimes [1,0], 1_n \otimes [0,1]\}$ might span the null space of $\mathcal{L}(\alpha^*) + B$. Note that $\{p^*, (I_n \otimes R(\frac{\pi}{2}))p^*\}$ are not in the null space of $[\mathcal{L}_{ll} + \bar{B}, \mathcal{L}_{lf}]$ because $\mathcal{L}_{ll}p_a^* + \mathcal{L}_{lf}p_f^* = 0$ and $\bar{B}p_a^* \neq 0$.

In conclusion, only $1_n \otimes [1,0]^\top$ and $1_n \otimes [0,1]^\top$ span the null space of $\mathcal{L}(\alpha^*) + B$, which implies that $\lim_{t\to\infty}(p(t) - \delta^* - \beta_5 1_n \otimes [1,0] - \beta_6 1_n \otimes [1,0]) = 0$, i.e., $\lim_{t\to\infty}(p_i(t) - p_j(t) - \delta_{ij}) = 0, \forall i,j \in \mathcal{V}$. □

Remark 3. *Compared to the nonlinear formation controllers proposed in [7], [19], [23], [27] to locally or almost globally achieve a desired formation with a prescribed scale or a presecribed orientation and scale, our proposed control law* (16) *is linear and guarantees global convergence. Compared to the affine formation approach where at least three leaders are needed to fix the formation's orientation and scale [24], only two leaders are needed in our approach. The convergent formation space of this angle-based formation is a subset of the corresponding affine formation since affine formation includes not only translation, rotation, scaling, but also sheering with respect to the nominal configuration.*

B. Agent 1 controls relative position with respect to agent 2

Note that when both agents 1 and 2 control their relative position, the orientations of agents 1's and 2's coordinate frames need to be aligned. To avoid the requirement on the alignment of the two agents' coordinate frames, we investigate the case that only agent 1 controls the relative position with respect to agent 2. We now propose the following control laws for all the agents

$$\begin{aligned} u_i(t) = -\Big[&\sum\nolimits_{(i,j_1,k_1)\in\bar{\mathcal{A}}} (A_i^{\triangle ij_1k_1}(\alpha^*))^\top f_i^{\triangle ij_1k_1}(\alpha^*, p(t)) \\ &+ \sum\nolimits_{(j_2,i,k_2)\in\bar{\mathcal{A}}} (A_i^{\triangle j_2ik_2}(\alpha^*))^\top f_i^{\triangle j_2ik_2}(\alpha^*, p(t)) \\ &+ \sum\nolimits_{(j_3,k_3,i)\in\bar{\mathcal{A}}} (A_i^{\triangle j_3k_3i}(\alpha^*))^\top f_i^{\triangle j_3k_3i}(\alpha^*, p(t))\Big] - c_i(t) \end{aligned} \tag{22}$$

where $c_i(t) = \begin{cases} k_2(p_1(t) - p_2(t) - \delta_{21}^*), & \text{if } i=1 \\ 0, & \text{otherwise} \end{cases}$ and $k_2 \in \mathbb{R}$ is a positive gain to be determined. Following (11), the closed-loop dynamics under (22) can be written as

$$\dot{p}(t) = -(\mathcal{L}(\alpha^*) + C)(p(t) - \delta^*) \tag{23}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2022.3151567, IEEE Transactions on Automatic Control

where

$$C = \begin{bmatrix} k_2 I_2 & -k_2 I_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \tag{24}$$

Since $[\mathcal{L}_{fl}, \mathcal{L}_{ff}]$ is still a sub-matrix of $\mathcal{L}(\alpha^*) + C$ and $\delta_1^* \neq \delta_2^*$, it follows from the proof of Theorem 2 that the null space of $\mathcal{L}(\alpha^*) + C$ is only spanned by the two vectors $1_n \otimes [1, 0]^\top$ and $1_n \otimes [0, 1]^\top$, i.e., $\mathcal{L}(\alpha^*) + C$ only has two zero eigenvalues. However, since $\mathcal{L}(\alpha^*) + C$ is asymmetric, the other eigenvalues in addition to the two zero eigenvalues may have negative real parts, which can make the system (23) unstable.

To avoid this case, we discuss how to design the control gain k_2 such that all the remaining eigenvalues of $\mathcal{L}(\alpha^*) + C$ have positive real parts. First, we separate the two zero eigenvalues from $\mathcal{L}(\alpha^*) + C$. Defining $P \in \mathbb{R}^{2n\times 2n}$ as an orthonormal matrix whose first two rows are $(1_n^\top \otimes I_2)/\sqrt{n}$, one has

$$P(\mathcal{L}(\alpha^*) + C)P^\top = \begin{bmatrix} 0_{2\times 2} & 0 \\ 0 & \bar{\mathcal{L}} + k_2 \bar{C} \end{bmatrix} \tag{25}$$

where $\bar{\mathcal{L}} \in \mathbb{R}^{(2n-2)\times(2n-2)}$, $\bar{C} \in \mathbb{R}^{(2n-2)\times(2n-2)}$. One has the global stability of (19) if all the eigenvalues of $\bar{\mathcal{L}} + k_2 \bar{C}$ have positive real parts, which depends on the design of the gain k_2. To obtain an acceptable k_2, we transfer it to the following optimization problem

$$\begin{aligned} &\max_{k_2} \quad \gamma_1 \\ &\text{subject to} \quad (\bar{\mathcal{L}} + k_2\bar{C})^\top P + P(\bar{\mathcal{L}} + k_2\bar{C}) - \gamma_1 I_{2n-2} > 0 \\ &\qquad\qquad\quad P > 0 \end{aligned} \tag{26}$$

Now, we summarize the analysis as the following results.

Proposition 1. *If the desired triangular angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p^*)$ is angle rigid and the optimization problem* (26) *has a feasible solution, then the control law* (22) *will globally stabilize the angle rigid formation with the prescribed orientation and scale, i.e.,* $\lim_{t\to\infty}(p_i(t) - p_j(t) - \delta_{ij}) = 0, \forall i, j \in \mathcal{V}$.

Remark 4. *In the proposed formation control law* (22)*, the orientation alignment on agents' coordinate frames is not required. Note that the optimization problem* (26) *might not have an accessible solution. If this case happens, more agents need to be involved to achieve the desired formation.*

V. Extension to Ratio of Distance-Constrained Formations

As an extension, we discuss how to globally stabilize a desired multi-agent formation described by ratio of distances. Different from [12], we define the signed ratio of distance $r_{kij} \in (-\infty, 0) \cup (0, +\infty)$ among agents k, i, j as

$$r_{kij} := \begin{cases} d_{ki}/d_{ij}, & \text{if } b_{ij}^\top b_{ik}^\perp \geq 0, \\ -d_{ki}/d_{ij}, & \text{otherwise}, \end{cases} \tag{27}$$

Then, in a non-degenerate triangle $\triangle ijk$, the values of the angle constraints can be obtained from the values of the ratio-of-distance constraints by the law of cosines

$$\alpha_{kij} = \begin{cases} \arccos(\frac{1}{2}(|r_{kij}| + |r_{jik}| - |r_{jki} r_{ikj}|)), & \text{if } r_{kij} > 0 \\ 2\pi - \arccos[\frac{1}{2}(|r_{kij}| + |r_{jik}| - |r_{jki} r_{ikj}|)], & \text{otherwise} \end{cases}$$

Inversely, the values of the ratio-of-distance constraints can be obtained from the values of the angle constraints by the law of sines

$$r_{kij} = \begin{cases} \sin\alpha_{kji}/\sin\alpha_{ikj}, & \text{if } 0 < \alpha_{kji} < \pi \\ -\sin\alpha_{kji}/\sin\alpha_{ikj}, & \text{otherwise} \end{cases} \tag{28}$$

This bidirectional and unique mapping implies that describing a triangular formation by signed ratio-of-distance constraints is equivalent to describing the formation by angle constraints.

Lemma 4. *The signed ratio-of-distance constraints $r_{ijk}, r_{jki}, r_{kij}$ in the triangle $\triangle ijk$ uniquely determine the angle constraints $\alpha_{ijk}, \alpha_{jki}, \alpha_{kij}$, and vice versa.*

Under Lemma 4, the angle constraints $f_{\mathcal{A}}(\alpha^*)$ describing the desired angle rigid formation can be transferred into the signed ratio-of-distance constraints. After that, following the same control design in Sections III and IV, triangularly rigid formations specified by ratio-of-distance constraints can be globally stabilized.

VI. Simulation Examples

This section presents two simulation examples to validate Theorems 1 and 2, respectively. Consider a formation consisting of eight agents and the desired formation shape consisting of 6 triangle constraints: $\triangle 124, \triangle 146, \triangle 467, \triangle 457, \triangle 578, \triangle 367$. A set of agents' positions to construct the desired angle rigid formation is given as $p_1^* = [-4.2; 1.3], p_2^* = [-2.3; 6.1], p_3^* = [0.5; -4.1], p_4^* = [-0.4; 1.5], p_5^* = [0.8; 1.3], p_6^* = [-2.1; 0.2], p_7^* = [0.1; -0.1], p_8^* = [2.3; 0.1]$. To demonstrate that the angle rigid formation control system is globally stable, we randomly choose the eight agents' initial positions.

For the first case, we assume that there is no leader in the formation. Under the formation control law (11), the formation trajectories and the evolution of angle errors are shown in the left and right sides of Fig. 1, respectively.

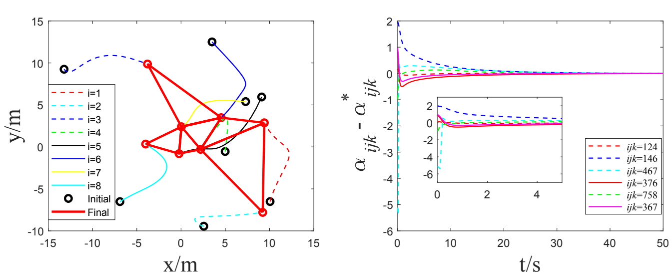

Fig. 1: Formation trajectories and evolution of angle errors

For the second case, we consider that agents 1 and 2 are leaders, and their desired relative position is $\delta_{12}^* = -\delta_{21}^* = 2(p_2^* - p_1^*) = [3.8; 9.6]$. Under the formation control law (16) with $k_1 = 1$, the evolution of angle errors and the evolution of relative position error between agents 1 and 2 are shown in the left and right sides of Fig. 2, respectively. The formation trajectories are shown in Fig. 3. The sudden change of $\alpha_{146}(t)$ from 2π to 0 at around $t = 4$s is because a collinearity occurs in $\triangle 146$, which however will not cause discontinuity on the formation evolution since the position dynamics (23) is continuous.

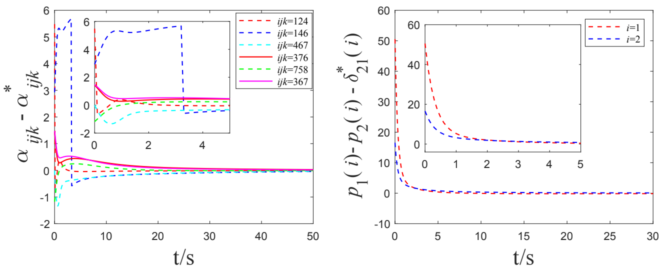

Fig. 2: Evolution of angle errors and evolution of relative position error between agents 1 and 2

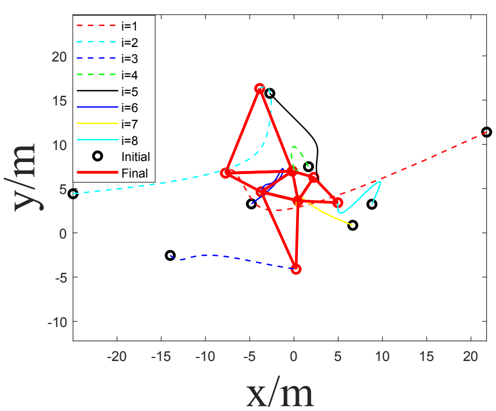

Fig. 3: Formation trajectories of all agents with the prescribed orientation and scale

VII. Conclusion

This paper has studied the global stabilization of triangularly angle rigid formations. Inspired by angle-induced linear constraints in triangles, we have proposed a formation control law to achieve the global stabilization of triangularly angle rigid formations using local relative position measurements. Our formation control approach has been shown to be computationally effective and scalable. Moreover, by additionally controlling the relative position between a pair of neighboring agents, modified formation control laws have been proposed to globally stabilize triangularly angle rigid formation with the prescribed orientation and scale. Future work will focus on the global stabilization of multi-agent systems governed by double-integrator dynamics in 3D space.

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