Angle Rigidity and Its Usage to Stabilize Multiagent Formations in 2-D

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***Abstract*—Motivated by the challenging formation stabilization problem for mobile robotic teams wherein no distance or relative position measurements are available but each robot can only measure some of relative angles with respect to its neighbors in its local coordinate frame, we develop the notion of "angle rigidity" for a multipoint framework, named "angularity", consisting of a set of nodes embedded in a Euclidean space, and a set of angle constraints among them. Different from bearings or angles defined in a global frame, the angles we use do not rely on the knowledge of a global frame, and are signed according to the counter-clockwise direction. Here, *angle rigidity* refers to the property specifying that under proper angle constraints, the angularity can only translate, rotate, or scale as a whole when one or more of its nodes are perturbed locally. We first demonstrate that this angle rigidity property, in sharp comparison to bearing rigidity or other reported rigidity related to angles of frameworks in the literature, is *not* a global property since an angle rigid angularity may allow flex ambiguity. We then construct necessary and sufficient conditions for *infinitesimal* angle rigidity by checking the rank of an angularity's rigidity matrix. We develop a combinatorial necessary condition for infinitesimal minimal angle rigidity. Using the developed theories, a formation stabilization algorithm is designed for a robotic team to achieve an angle rigid formation, in which only angle measurements are needed. Simulation examples demonstrate the advantages of the proposed angle-only formation control approach.**

***Index Terms*—Angle/bearing measurements, angle rigidity, formation control, angularity, multiagent systems.**

I. Introduction

OVER the past decades, *distance rigidity* has been intensively investigated both as a mathematical topic in graph theory [1], [2] and an engineering problem in applications including formations of multiagent systems [3], mechanical structures [4], and biological materials [5]. Distance rigidity [6] is defined using the property of distance preservation of translational and rotational motions of a multipoint framework. To determine whether a given framework is distance rigid, two methods have been reported. The first is to test the rank of the distance rigidity matrix which is derived from the infinitesimally distance rigid motions [7]. The second is enabled by Laman's theorem, which is a combinatorial test and works only for generic frameworks. More recently, *bearing rigidity* has been investigated, in which the shape of a framework is prescribed by the interpoint bearings or directions [8], [9]. By defining the bearing as a unit vector in a given global coordinate frame, bearing rigidity can be defined accordingly [9], [10]. To check whether a framework is bearing rigid, conditions similar to those for distance rigidity have been discussed [9]–[12].

Distance constraints in determining distance rigidity are in general quadratic in the associated end points' positions. While a bearing constraint is always linear in the associated end point's position, the description of bearings directly depends on the necessity of a global coordinate frame or a coordinate frame in $\mathrm{SE}(2)$ or $\mathrm{SE}(3)$ [13], [14]. Different from distance and bearing rigidity,[1] in this study, we aim at presenting *angle rigidity* theory for multipoint frameworks accommodating angle constraints as either linear or quadratic constraints on the end points' positions without relying on a global coordinate frame. Different from the usual definition for a scalar angle [15], [16], the angle defined in this article is signed. By defining the counter-clockwise direction to be each angle's positive direction, angle rigidity is defined for an *angularity* which consists of vertices and angle constraints among them. We show that the planar angle rigidity is a local property because of the existence of flex ambiguity. To check whether an angularity is angle rigid, angle rigidity matrix is derived based on the infinitesimally angle rigid motions. Then, the angle rigidity of an angularity can be determined by testing the rank of its angle rigidity matrix. Also, we develop a combinatorial necessary condition to test the angle rigidity of an angularity. We underline that the Laman's theorem and Henneberg's construction method do not apply directly to angle rigidity, which makes our results crucial. Using the defined signed angles, we further propose the construction methods for angle rigid and globally angle rigid angularities.

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[1]Only refer to bearing rigidity in $\mathbb{R}^2$ in the following.

Besides its mathematical importance, angle rigidity is closely related to the application in multiagent formation control for robotic transportation [17], search and rescue of drones [18], and satellite formation flying [19]. Sensors used in formation stabilization mainly include GPS receivers, radars, and cameras, which can acquire absolute positions, interagent relative positions, or angles/bearings [3], [20]. In particular, angle measurements are becoming cheaper, more reliable, and accessible than absolute or relative position measurements [12], [20]. Angle information can be easily obtained by a passive sonar, camera, or sensor array in its local coordinate frame [17]. Using angle rigidity developed in this article, we show how to stabilize a planar formation by using only local angle measurements. Different from the designed bearing-based formation control algorithms in [9], [21] where all agents' local coordinate frames are required to be aligned, the proposed angle-only formation control algorithm does not require the alignment of agents' coordinate frames since the angle described in different planar coordinate frames remains the same. We acknowledge that in [15], planar angle rigidity is established by employing the cosine of an angle formed by two joint edges as the angle constraint. The formation stabilization algorithm constructed in [15] requires that each agent can sense the relative positions with respect to its neighbors. Different from [15], in this article, the desired formation shape is realized using only angle measurements. In addition, weak rigidity with mixed distance and angle constraints has been investigated in [22]–[24], under which the formation control algorithms are also designed for agents by using the measurements of relative position.

The rest of this article is organized as follows. Section II gives the definition of an angularity and its rigidity. Section III introduces infinitesimal angle rigidity. In Section IV, the application in multiagent planar formations is investigated. Simulation examples are provided in Section V.

II. Angularity and Its Angle Rigidity

Graphs have been used dominantly in rigidity theory for multipoint frameworks under distance constraints since an edge of a graph can be naturally used to denote the existence of a distance constraint between the two points corresponding to the vertices adjacent to this edge. However, when describing angles formed by rays connecting points, to use edges of a graph becomes inappropriate because an angle constraint always involves three points. For this reason, instead of using graphs that relate pairs of vertices as the main tool to define rigidity, we define a new combinatorial structure "angularity" that relates triples of vertices to develop the theory of angle rigidity. In all the following discussions, we confine ourselves to the plane.

A. Angularity

We use the vertex set $\mathcal{V} = \{1, 2, \dots, N\}$ to denote the set of indices of the $N \geq 3$ points of a framework in the plane. As shown in Fig. 1, to describe the *signed* angle from the ray $j - i$ to ray $j - k$, one needs to use the ordered triplet (i, j, k), and, obviously, the two angles corresponding to (i, j, k) and (k, j, i)

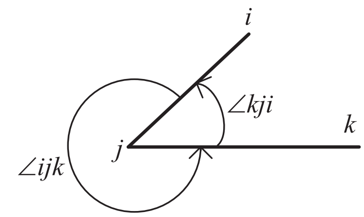

Fig. 1. Signed angle used in defining angle rigidity.

are different, and in fact are called explementary or conjugate angles. Here, following convention, the angle $\measuredangle ijk$ for each triplet (i, j, k) is measured counter-clockwise in the range $[0, 2\pi)$. We use $\mathcal{A} \subset \mathcal{V} \times \mathcal{V} \times \mathcal{V} = \{(i, j, k), i, j, k \in \mathcal{V}, i \neq j \neq k\}$ to denote the *angle set*, each element of which is an ordered triplet. We denote the number of elements $|\mathcal{A}|$ of the angle set $\mathcal{A}$ by M. Throughout this article, we assume that no pair of triplets in $\mathcal{A}$ are explementary to each other. Now consider the embedding of the vertex set $\mathcal{V}$ in the plane $\mathbb{R}^2$ through which each vertex i is associated with a distinct position $p_i \in \mathbb{R}^2$ and let $p = [p_1^{\mathrm{T}}, \dots, p_N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2N}$. We assume there is no overlapping points in p, i.e., $p_i \neq p_j$ for $i \neq j$ and $i, j \in \{1, 2, \dots, N\}$. Then, the combination of the vertex set $\mathcal{V}$, the angle set $\mathcal{A}$, and the position vector p is called an *angularity*, which we denote by $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$. Actually, given nonoverlapping positions p_i, p_j, and p_k, the angle $\measuredangle ijk \in [0, 2\pi)$ can be uniquely calculated from

$$\measuredangle ijk = \begin{cases} \arccos(z_{ji}^{\mathrm{T}} z_{jk}) & \text{if } z_{ji}^{\perp} \cdot z_{jk} \geq 0 \\ 2\pi - \arccos(z_{ji}^{\mathrm{T}} z_{jk}) & \text{otherwise} \end{cases} \quad (1)$$

where $z_{ji} = \frac{p_i - p_j}{\|p_i - p_j\|}$, $z_{jk} = \frac{p_k - p_j}{\|p_k - p_j\|}$, $z_{ji}^{\perp} = Q_0 z_{ji} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} z_{ji}$ is the vector obtained by rotating z_{ji} counter-clockwise by $\frac{\pi}{2}$, and $\cdot$ denotes the dot product.

B. Angle Rigidity

We first define what we mean by two equivalent or congruent angularities.

Definition 1 (Equivalency and congruency): We say two angularities $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ and $\mathbb{A}'(\mathcal{V}, \mathcal{A}, p')$ with the same $\mathcal{V}$ and $\mathcal{A}$ are *equivalent* if

$$\measuredangle ijk(p_i, p_j, p_k) = \measuredangle ijk(p_i', p_j', p_k') \text{ for all } (i, j, k) \in \mathcal{A}. \quad (2)$$

We say they are *congruent* if

$$\measuredangle ijk(p_i, p_j, p_k) = \measuredangle ijk(p_i', p_j', p_k') \text{ for all } i, j, k \in \mathcal{V}. \quad (3)$$

From the equivalent and congruent relationships, it is easy to define global angle rigidity.

Definition 2 (Global angle rigidity): An angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is *globally angle rigid* if every angularity that is equivalent to it is also congruent to it.

When such a rigidity property holds only locally, one has angle rigidity.

Definition 3 (Angle rigidity): An angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is *angle rigid* if there exists an $\epsilon > 0$ such that every angularity $\mathbb{A}'(\mathcal{V}, \mathcal{A}, p')$ that is equivalent to it and satisfies $\|p' - p\| < \epsilon$ is congruent to it.

Definition 3 implies that every configuration which is sufficiently close to p and satisfies all the angle constraints formed

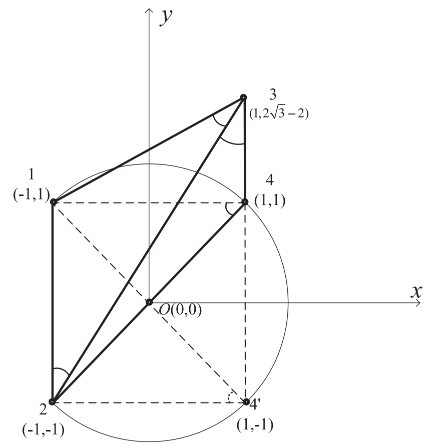

Fig. 2. Flex ambiguity in angle rigid angularity.

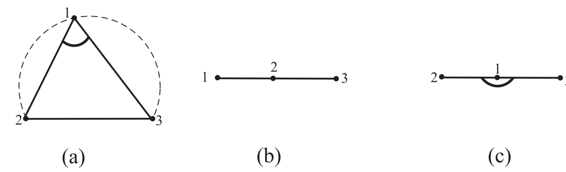

Fig. 3. Nongeneric p changes rigidity. (a) Not rigid when $\measuredangle 213 = \frac{\pi}{3}$. (b) Angle rigid when $\measuredangle 213 = 0$. (c) Globally angle rigid when $\measuredangle 213 = \pi$.

by $\mathcal{A}$ has the same magnitudes of the angles formed by any three vertices in $\mathcal{V}$ as the original configuration at p.

As is clear from Definitions 2 and 3, global angle rigidity always implies angle rigidity. A natural question to ask is whether angle rigidity also implies global angle rigidity. In fact, for bearing rigidity, it has been shown that indeed global bearing rigidity and bearing rigidity are equivalent [9], [10]. However, this is *not* the case for angle rigidity.

Theorem 1 (Nonequivalence between angle rigidity and global angle rigidity): An angle rigid angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is not necessarily globally angle rigid.

We prove this theorem by providing the following example. Fig. 2 shows an angularity with $\mathcal{V} = \{1, 2, 3, 4\}$, and its elements in the set $\mathcal{A} = \{(3, 2, 1), (1, 3, 2), (2, 3, 4), (1, 4, 2)\}$ take the values

$$\measuredangle 321 = \arccos\left(\frac{4\sqrt{3}-2}{2\sqrt{17-4\sqrt{3}}}\right) \approx 39.07^\circ \tag{4}$$

$$\measuredangle 132 = \arccos\left(\frac{19-8\sqrt{3}}{\sqrt{25-12\sqrt{3}}\sqrt{17-4\sqrt{3}}}\right) \approx 37.88^\circ \tag{5}$$

$$\measuredangle 234 = 30^\circ \tag{6}$$

$$\measuredangle 142 = 45^\circ \tag{7}$$

and its p is shown as in the coordinates of the vertices. We first show $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is angle rigid, and then show $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is not globally angle rigid.

Now first look at the triangle formed by 1, 2, and 3. Since two of its angles $\measuredangle 321$ and $\measuredangle 132$ have been constrained, the remaining $\measuredangle 213$ is uniquely determined to be $\pi - \measuredangle 321 - \measuredangle 132$, no matter how p is locally perturbed. The constraint on $\measuredangle 234$ requires 4 must lie in the ray starting from 3 and rotating from ray 32 counter-clockwise by 30°; at the same time, the constraint on $\measuredangle 142$ requires 4 must lie on the arc passing through 1 and 2 such that the inscribed angle $\measuredangle 142$ is 45°. No matter how p is locally perturbed, there is only one unique position for 4 in the neighborhood of its current given coordinates because the two intersection points between the ray and the arc are not in the same local neighborhood. This local uniqueness implies that this four-vertex angularity is angle rigid (when 4's position is uniquely determined, any angle associated with it is also uniquely determined).

Now we show $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is not globally angle rigid. Note that there is the other intersection point $4'$ as shown in Fig. 2 satisfying the angle constraints given in $\mathcal{A}$, which implies that this angularity is not globally angle rigid because $\mathbb{A}(\mathcal{V}, \mathcal{A}, p')$ is equivalent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, but they are not congruent.

We provide the following further insight to explain this sharp difference between the angle rigidity that we have defined and the bearing rigidity that has been reported in the literature. Bearing rigidity as defined in [9], [10] is a global property because the bearing constraints always represent linear constraints in the end point's position (similar to the angle constraint $\measuredangle 234 = 30^\circ$ in the form of the ray from 3 to 4 in the above example) and two noncollinear rays have at most one intersection. In contrast, our angle constraint can be either linear constraint in p when it requires the corresponding vertex to be on a ray or quadratic in p when it restricts the corresponding vertex to be on an arc passing through other vertices. The possible nonlinearity in the angle constraints gives rise to potential ambiguity of the vertices' positions under the given angle constraints.

Note that the embedding of p in the plane may affect the rigidity of $\mathbb{A}$. Consider the 3-vertex angularity as embedded in three different situations in Fig. 3 when its angle set $\mathcal{A}$ contains only one element (2,1,3). Fig. 3(a) shows that 1, 2, and 3 are not collinear, and then this angularity is in general not rigid since if we perturb point 1 in an arc with 2 and 3 as the arc's ending points, $\measuredangle 213$ can be the same while angles $\measuredangle 123$ and $\measuredangle 132$ change. In Fig. 3(b), 1, 2, and 3 are collinear and 1 is on one side; in this case, if the angle constraint happens to be $\measuredangle 213 = 0$, then one can check the angularity becomes angle rigid, although it is not globally rigid since the angle of $\measuredangle 132$ changes by 180 degree if we swap 2 and 3. In Fig. 3(c), 1, 2, and 3 are collinear and 1 is in the middle; when the constraint becomes $\measuredangle 213 = \pi$, one can check that the angularity is not only rigid, but also globally rigid (swapping of 2 and 3 in this case does not change the resulting angles $\measuredangle 132, \measuredangle 123$ being zero). So the angularity $\mathbb{A}(\{1, 2, 3\}, \{(2, 1, 3)\}, p)$ is generically not rigid, but rarely rigid depending on p. To clearly describe this relationship between angle rigidity and p, like in standard rigidity theory, we define what we mean by generic positions.

Definition 4 (Generic position): The position vector p is said to be *generic* if its components are algebraically independent [25]. Then, we say an angularity is *generically (respectively globally) angle rigid* if its p is generic and it is (respectively globally) angle rigid.

An example for nongeneric positions is the case when three points are collinearly positioned. Note that angle rigidity for

$\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with generic p represents the common property of the combination $(\mathcal{V}, \mathcal{A})$ from a topological perspective, which is also referred to as *generic angle rigidity*. For convenience, we also say an angularity is generic if its p is generic. Now we provide some sufficient conditions for an angularity to be globally angle rigid. Towards this end, we need to introduce some concepts and operations. For two angularities $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ and $\mathbb{A}'(\mathcal{V},' \mathcal{A},' p')$, we say $\mathbb{A}$ is a *subangularity* of $\mathbb{A}'$ if $\mathcal{V} \subset \mathcal{V}'$, $\mathcal{A} \subset \mathcal{A}'$ and p is the corresponding subvector of p'. We first clarify that for the smallest angularities, namely those that contain only three vertices, there is no gap between angle rigidity and global angle rigidity assuming generic positions.

Lemma 1: For a 3-vertex angularity, if it is generically angle rigid, it is also generically globally angle rigid.

Proof: For this 3-vertex angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, since it is angle rigid and p is generic, $\mathcal{A}$ must contain at least two elements, or said differently, two of the interior angles of the triangle formed by the three vertices are constrained. Again since p is generic, the sum of the three interior angles in this triangle has to be π, and thus the magnitude of this triangle's remaining interior angle is uniquely determined too. Therefore, $\mathbb{A}$ is generically globally angle rigid. ■

Now, we define linear and quadratic constraints.

Definition 5 (Linear and quadratic constraints): For a given angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, a new vertex i positioned at p_i is *linearly constrained* with respect to $\mathbb{A}$ if there is $j \in \mathcal{V}$ such that $p_i \neq p_j$ and p_i is constrained to be on a ray starting from p_j; we also say i is *quadratically constrained* with respect to $\mathbb{A}$ if there are $j, k \in \mathcal{V}$ such that $\{p_i, p_j, p_k\}$ is generic and p_i is constrained to be on an arc with p_j and p_k being the arc's two ending points. Correspondingly, we call i's constraint in the former case a *linear constraint* and in the latter case a *quadratic constraint* with respect to $\mathbb{A}$.

As shown in Fig. 2, $\measuredangle 234 = 30^\circ$ is a linear constraint for the end vertex 4 since p_4 is constrained to be on a ray starting from p_3, while $\measuredangle 142 = 45^\circ$ is a quadratic constraint for 4 because p_4 is constrained to be on the major arc $\overset{\frown}{12}$.

Similar to Henneberg's construction in distance rigidity, in the following, we define two types of vertex addition operations in angle rigidity to demonstrate how a bigger angularity might grow from a smaller one, which are shown in Fig. 4.

Definition 6 (Type-I vertex addition): For a given angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, we say the angularity $\mathbb{A}'$ with the augmented vertex set $\{\mathcal{V} \cup \{i\}\}$ is obtained from $\mathbb{A}$ through a *Type-I vertex addition* if the new vertex i's constraints with respect to $\mathbb{A}$ contain at least one of the following.

Case 1) two linear constraints, not aligned, associated with two distinct vertices in $\mathcal{V}$ (one vertex for one constraint and the other vertex for the other constraint).

Case 2) one linear constraint and one quadratic constraint associated with two distinct vertices in $\mathcal{V}$ (one for the former and both for the latter).

Case 3) two different quadratic constraints associated with three vertices in $\mathcal{V}$ (two for each and one is shared by both), and the positions of i and these three vertices are generic.

Definition 7 (Type-II vertex addition): For a given angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, we say the angularity $\mathbb{A}'$ with the augmented vertex set $\{\mathcal{V} \cup \{i\}\}$ is obtained from $\mathbb{A}$ through a *Type-II vertex addition* if the new vertex i's constraints with respect to $\mathbb{A}$ contain at least one of the following:

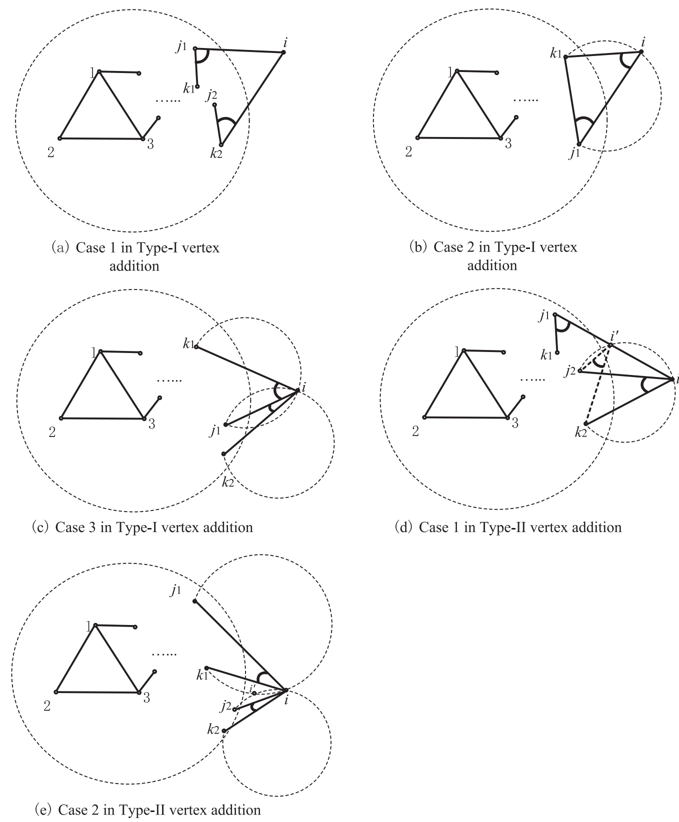

Fig. 4. Type-I vertex addition and Type-II vertex addition. (a) Case 1 in Type-I vertex addition. (b) Case 2 in Type-I vertex addition. (c) Case 3 in Type-I vertex addition. (d) Case 1 in Type-II vertex addition. (e) Case 2 in Type-II vertex addition.

Case 1) one linear constraint and one quadratic constraint associated with three distinct vertices in $\mathcal{V}$ (one for the former and the other two for the latter);

Case 2) two different quadratic constraints associated with four vertices in $\mathcal{V}$ (two for the former and the other two for the latter), and the positions of i and these four vertices are generic.

Remark 1: Although the types of constraints are similar between Case 2 of Definition 6 and Case 1 of Definition 7, the numbers of vertices involved in Case 2 of Definition 6 and Case 1 of Definition 7 differ in these two types of vertex addition operations. Similarly, those in Case 3 of Definition 6 and Case 2 of Definition 7 are also different.

Remark 2: Note that in these two vertex addition operations, the involved vertices are required to be in generic positions. However, the overall angle rigid angularity $\mathbb{A}'$ constructed through a sequence of vertex addition operations is *not* necessarily generic, and an example is given in Fig. 5.

Now we are ready to present a sufficient condition for global angle rigidity using Type-I vertex addition.

Proposition 1 (Sufficient condition for global angle rigidity): An angularity is globally angle rigid if it can be obtained through a sequence of Type-I vertex additions from a generically angle rigid 3-vertex angularity.

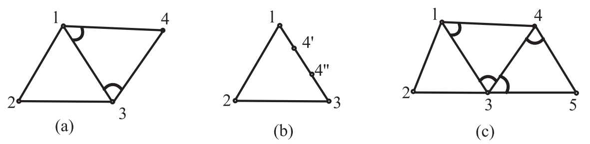

Fig. 5. Overall angularity is not necessarily generic. (a) Point 4 is unique when {1,3,4} are generic. (b) Point 4 is not unique when {1,3,4} are not generic. (c) {2,3,5} are not generic but angularity is rigid.

Proof: According to Lemma 1, the generically angle rigid 3-vertex angularity is globally angle rigid. Consider the three conditions in the Type-I vertex addition. If case 1 applies, then the position p_i of the newly added vertex i is unique since two rays, not aligned, starting from two different points may intersect only at one point; if case 2 applies, p_i is again unique since a ray starting from the end point of an arc may intersect with the arc at most at one other point; and if case 3 applies, p_i is unique since two arc sharing one end point on different circles can only intersect at most at one other point. Therefore, p_i is always globally uniquely determined. After p_i is globally uniquely determined, all the angles associated with p_i are also globally uniquely determined. Because each Type-I vertex addition operation can guarantee a unique adding point p_i, we conclude that the obtained angularity after a sequence of Type-I vertex additions is globally angle rigid. ■

In comparison, Type-II vertex additions can only guarantee angle rigidity, but not global angle rigidity.

Proposition 2 (Sufficient condition for angle rigidity): An angularity is angle rigid if it can be obtained through a sequence of Type-II vertex additions from a generically angle rigid 3-vertex angularity.

The proof can be easily constructed following similar arguments as those for Proposition 1. The only difference is that p_i now may have two solutions and is only unique locally.

After having presented our results on angularity and angle rigidity, in the following section, we discuss infinitesimal angle rigidity, which relates closely to infinitesimal motion.

III. Infinitesimal Angle Rigidity

Analogous to distance rigidity, infinitesimal angle rigidity can be characterized by the kernel of a properly defined rigidity matrix. Towards this end, we first introduce the following angle function. For each angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, we define the *angle function* $f_{\mathcal{A}}(p) : \mathbb{R}^{2N} \to \mathbb{R}^M$ by

$$f_{\mathcal{A}}(p) := [f_1, \ldots, f_M]^{\mathrm{T}} \tag{8}$$

where $f_m : \mathbb{R}^6 \to [0, 2\pi)$, $m = 1, \ldots, M$, is the mapping from the position vector $[p_i^{\mathrm{T}}, p_j^{\mathrm{T}}, p_k^{\mathrm{T}}]^{\mathrm{T}}$ of the mth element (i, j, k) in $\mathcal{A}$ to the signed angle $\measuredangle ijk \in [0, 2\pi)$. Using this angle function, one can define $\mathbb{A}$'s angle rigidity matrix.

A. Angle Rigidity Matrix

We consider an arbitrary element (i, j, k) in $\mathbb{A}$ and denote the corresponding angle constraint by $\measuredangle ijk(p_i, p_j, p_k) = \beta \in [0, 2\pi)$, or in shorthand $\measuredangle ijk = \beta$. From the definition of the dot product, one has

$$\cos\beta = \frac{(p_i - p_j)^{\mathrm{T}}}{\|p_i - p_j\|} \frac{(p_k - p_j)}{\|p_k - p_j\|} = z_{ji}^{\mathrm{T}} z_{jk} \tag{9}$$

where $\|\cdot\|$ denotes the Euclidean vector norm and we have used $\cos\beta = \cos(2\pi - \beta)$ according to (1). Differentiating both sides of (9) with respect to time leads to

$$\begin{aligned}(-\sin\beta)\dot{\beta} &= \dot{z}_{ji}^{\mathrm{T}} z_{jk} + z_{ji}^{\mathrm{T}} \dot{z}_{jk} \\ &= \left[\frac{P_{z_{ji}}}{l_{ji}}(\dot{p}_i - \dot{p}_j)\right]^{\mathrm{T}} z_{jk} + z_{ji}^{\mathrm{T}} \frac{P_{z_{jk}}}{l_{jk}}(\dot{p}_k - \dot{p}_j)\end{aligned} \tag{10}$$

where $l_{jk} = \|p_j - p_k\|$, $P_{z_{ji}} = I_2 - z_{ji} z_{ji}^{\mathrm{T}}$, I_2 denotes the 2×2 identity matrix, and we have used the fact that for $x \in \mathbb{R}^2, x \neq 0$, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{\|x\|}\right) = \frac{P_{x/\|x\|}}{\|x\|}\dot{x}$. By rearranging (10), one obtains

$$\begin{aligned}\frac{\mathrm{d}\beta}{\mathrm{d}t} &= \frac{\partial\beta}{\partial p_i}\dot{p}_i + \frac{\partial\beta}{\partial p_j}\dot{p}_j + \frac{\partial\beta}{\partial p_k}\dot{p}_k \\ &= N_{kji}\dot{p}_i - (N_{kji} + N_{ijk})\dot{p}_j + N_{ijk}\dot{p}_k\end{aligned} \tag{11}$$

where $N_{kji} = -\frac{z_{jk}^{\mathrm{T}} P_{z_{ji}}}{l_{ji}\sin\beta} \in \mathbb{R}^{1\times 2}$, $N_{ijk} = -\frac{z_{ji}^{\mathrm{T}} P_{z_{jk}}}{l_{jk}\sin\beta} \in \mathbb{R}^{1\times 2}$, and we have assumed $\sin\beta \neq 0$, i.e., no collinearity among p_i, p_j, and p_k. For each (i, j, k) in $\mathcal{A}$, we obtain an equation in the form of (11), and then one can write such M equations into the matrix form

$$\frac{\mathrm{d}f_{\mathcal{A}}(p)}{\mathrm{d}t} = \frac{\partial f_{\mathcal{A}}(p)}{\partial p}\dot{p} = R_a(p)\dot{p} \tag{12}$$

where $R_a(p) \in \mathbb{R}^{M\times 2N}$ is called the *angle rigidity matrix*, whose rows are indexed by the elements of $\mathcal{A}$ and columns the coordinates of the vertices

$$R_a(p) = \frac{\partial f_{\mathcal{A}}(p)}{\partial p} = \tag{13}$$

$$\begin{array}{c} \\ \text{Angle } 1 \\ \cdots \\ \measuredangle ijk \\ \cdots \\ \text{Angle } M \end{array} \begin{array}{c} \begin{array}{ccccccc} \cdots & \text{Vertex } i & \cdots & \text{Vertex } j & \cdots & \text{Vertex } k & \cdots \end{array} \\ \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & N_{kji} & 0 & -N_{kji} - N_{ijk} & 0 & N_{ijk} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \end{array}$$

For an angularity, its angle preservation motions satisfy $\dot{f}_{\mathcal{A}} = R_a(p)\dot{p} = 0$ which include translation, rotation, and scaling. One may rightfully expect that such motions are captured by the null space of the angle rigidity matrix, which always contains the following four linearly independent vectors:

$$q_1 = 1_N \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; q_2 = 1_N \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{14}$$

$$q_3 = \left[(Q_0 p_1)^{\mathrm{T}}, \quad (Q_0 p_2)^{\mathrm{T}}, \quad \ldots, \quad (Q_0 p_N)^{\mathrm{T}}\right]^{\mathrm{T}} \tag{15}$$

$$q_4 = \left[(\kappa p_1)^{\mathrm{T}}, \quad (\kappa p_2)^{\mathrm{T}}, \quad \ldots, \quad (\kappa p_N)^{\mathrm{T}}\right]^{\mathrm{T}} \tag{16}$$

where $\kappa \in \mathbb{R}$ is a nonzero scaling factor, $\otimes$ represents the Kronecker product, and 1_N denotes the $N \times 1$ column vector of all ones. Note that q_1 and q_2 correspond to translation, q_3 rotation, and q_4 scaling. We state this fact as a lemma.

Lemma 2 (Rank of angle rigidity matrix): For an angle rigidity matrix $R_a(p)$, it always holds that $\mathrm{Span}\{q_1, q_2, q_3, q_4\} \subseteq \mathrm{Null}(R_a(p))$ and correspondingly $\mathrm{Rank}(R_a(p)) \leq 2N - 4$.

Proof: Because each row sum of $R_a(p)$ equals zero, one has $R_a(p)q_1 = 0$ and $R_a(p)q_2 = 0$. Taking an arbitrary row $\measuredangle ijk$ in $R_a(p)$ as an example, one has the corresponding row in $R_a(p)q_3$

$$\begin{aligned} N_{kji}Q_0(p_i - p_j) &+ N_{ijk}Q_0(p_k - p_j) \\ &= \frac{z_{jk}^{\mathrm{T}} P_{z_{ji}} Q_0 z_{ji} + z_{ji}^{\mathrm{T}} P_{z_{jk}} Q_0 z_{jk}}{-\sin\beta} \\ &= \frac{z_{jk}^{\mathrm{T}} Q_0 z_{ji} + z_{ji}^{\mathrm{T}} Q_0 z_{jk}}{-\sin\beta} = 0 \end{aligned} \tag{17}$$

where we have used the fact that $Q_0^{\mathrm{T}} = -Q_0$ and $z_{ji}^{\mathrm{T}} Q_0 z_{ji} = 0$. Similarly, for $R_a(p)q_4$, one has

$$\begin{aligned} \kappa N_{kji}(p_i - p_j) &+ \kappa N_{ijk}(p_k - p_j) \\ &= \kappa \frac{z_{jk}^{\mathrm{T}} P_{z_{ji}} z_{ji} + z_{ji}^{\mathrm{T}} P_{z_{jk}} z_{jk}}{-\sin\beta} = 0 \end{aligned} \tag{18}$$

where we have used the fact that $P_{z_{ji}} z_{ji} = 0$. Therefore, $\mathrm{Span}\{q_1, q_2, q_3, q_4\} \subseteq \mathrm{Null}(R_a(p))$.

Since p has no overlapping elements, one has that q_3 and q_4 are linearly independent to q_1 and q_2. Because $q_1^{\mathrm{T}} q_2 = 0$ and $q_3^{\mathrm{T}} q_4 = 0$, one has that q_1, q_2, q_3, and q_4 are linearly independent. ■

Obviously the row rank of the angle rigidity matrix, or equivalently its row linear dependency, is a critical property of an angularity. We capture this property by using the notion of "independent" angles.

Definition 8 (Independent angles): For an angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, we say its angles in $f_{\mathcal{A}}(p)$ are *independent* if its angle rigidity matrix $R_a(p)$ has full row rank.

Since rank is a generic property of a rigidity matrix, one may wonder whether it is possible to disregard p of $\mathbb{A}$ and check generic angle rigidity only using $(\mathcal{V}, \mathcal{A})$. This is indeed doable as what we will show in the following subsection. Note that $2N - 4$ is the maximum rank that $R_a(p)$ can have. When p is generic, the exact realization of p is not important for $(\mathcal{V}, \mathcal{A})$, and when checking the angle rigidity matrix's rank, one can replace p by a random generic realization.

Using the notion of infinitesimal motion, checking the rank of the rigidity matrix can also enable us to check "infinitesimal" angle rigidity.

B. Infinitesimal Angle Rigidity

To consider infinitesimal motion, suppose that each $p_i, \forall i \in \mathcal{V}$ of $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is on a differentiable smooth path. We say the whole path $p(t)$ is generated by an *infinitesimally angle rigid motion* of $\mathbb{A}$ if on the path $f_{\mathcal{A}}(p)$ remains constant. We say such an infinitesimally angle rigid motion $p(t)$ is *trivial* if it can be given by [26]

$$p_i(t) = \kappa(t)\mathcal{Q}(t)p_i(t_0) + \mathcal{W}(t), \forall i \in \mathcal{V}, t \geq t_0 \tag{19}$$

where $\kappa(t) \neq 0$ is a scalar scaling factor, $\mathcal{Q}(t) \in \mathbb{R}^{2\times 2}$ is a rotation matrix, $\mathcal{W}(t) \in \mathbb{R}^2$ is a translation vector, and $\kappa(t), \mathcal{Q}(t), \mathcal{W}(t)$ are all differentiable smooth functions. Since all $p_i(t), \forall i \in \mathcal{V}$, share the same $\kappa(t), \mathcal{Q}(t), \mathcal{W}(t)$, it follows:

$$p(t) = \{I_N \otimes [\kappa(t)\mathcal{Q}(t)]\}p(t_0) + 1_N \otimes \mathcal{W}(t), t \geq t_0. \tag{20}$$

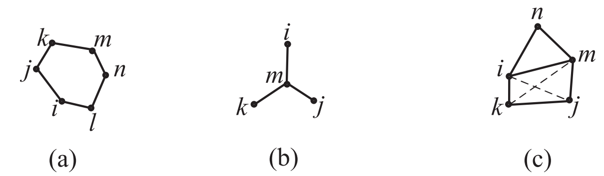

Fig. 6. Types of dependent triplet elements in $\mathcal{A}$. (a) Cycle. (b) Triplets with the same middle vertex. (c) Overly constrained angle subset.

Now we are ready to define infinitesimal angle rigidity.

Definition 9 (Infinitesimal angle rigidity): An angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is *infinitesimally angle rigid* if all its continuous infinitesimally angle rigid motion $p(t)$ are trivial.

In fact, a motion satisfying (20) is always an infinitesimally angle rigid motion because the combination of translation, rotation, and scaling preserves all the angle constraints. However, the converse does not necessarily hold, e.g., nontrivial infinitesimally angle rigid motion exists when only point 1 moves along line 12 in Fig. 3(b). We formalize these remarks in the following theorem.

Theorem 2 (Sufficient and necessary condition for infinitesimal angle rigidity): An angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is infinitesimally angle rigid if and only if the rank of its angle rigidity matrix $R_a(p)$ is $2N - 4$.

Proof: In view of the definition, $\mathbb{A}$ is infinitesimally angle rigid if and only if all its infinitesimally angle rigid motions are trivial. That is to say, these infinitesimally angle rigid motions $p(t), t \in [t_0, t_1]$ maintaining the angle constraints are exactly the combination of translation, rotation, and scaling with respect to the initial configuration $p(t_0)$, which are precisely captured by the four linearly independent vectors q_1, q_2, q_3, and q_4, which in turn is equivalent to the fact that the rigidity matrix's null space is precisely the span of $\{q_1, q_2, q_3, q_4\}$. The conclusion then follows from the fact that such a specification of the null space holds if and only if the rank of the rigidity matrix reaches its maximum $2N - 4$. ■

Note that this theorem implies that $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is infinitesimally angle rigid if and only if there are $2N - 4$ independent angles in $f_{\mathcal{A}}(p)$. We want to further remark that no matter what p is, if one of the following three combinatorial structures appears in $\mathcal{A}$, then the angles are always dependent.

1) A cycle formed by the triplets in $\mathcal{A}$ and $M = N$. For example, $\mathcal{A} = \{(i, j, k), (j, k, m), (k, m, n), (m, n, l), (n, l, i), (l, i, j)\}$ [see Fig. 6(a)].

2) Triplets with the same middle vertex and $M = N - 1$. For example, $\mathcal{A} = \{(i, m, j), (j, m, k), (k, m, i)\}$ [see Fig. 6(b)].

3) An angle subset $\mathcal{A}' \subset \mathcal{A}$ such that the number N' of the involved vertices in $\mathcal{A}'$ satisfies $|\mathcal{A}'| > 2N' - 4$. For example, $\mathcal{A} = \{(i, m, j), (m, j, i), (i, k, j), (i, j, k), (k, m, j), (n, i, m), (n, m, i)\}$ and $\mathcal{A}' = \{(i, m, j), (m, j, i), (i, k, j), (i, j, k), (k, m, j)\}$, and thus $N' = 4$, $|\mathcal{A}'| = 5$ in Fig. 6(c).

If $\mathcal{A}$ contains one of the above three combinatorial structures, we say the triplet elements in $\mathcal{A}$ are dependent; otherwise, they are independent. One can further quantify the number of triplet elements such that the angularity is infinitesimally angle rigid.

Theorem 3 (Combinatorial necessary condition for infinitesimal angle rigidity): For an angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, if it is infinitesimally angle rigid, then it has $2N-4$ independent triplet elements in $\mathcal{A}$.

Proof: From Theorem 2, we know $\mathbb{A}$ has $2N-4$ independent angles in $f_{\mathcal{A}}(p)$. First, we prove that dependent triplet elements in $\mathcal{A}$ imply dependent angles in $f_{\mathcal{A}}(p)$. Using geometric transformation, one has $N_{kji}^{\mathrm{T}} = \frac{(l_{jk}\cos\measuredangle ijk)z_{ji}-(p_k-p_j)}{l_{ji}l_{jk}\sin\measuredangle ijk} = -\frac{(p_i-p_j)^{\perp}}{l_{ij}^2}$. Then, by taking the dependent triplet elements in Fig. 6(a) as an example, it can be verified that

$$\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix} R_a(p) = 0 \tag{21}$$

which implies the row dependence in $R_a(p)$ and dependent angles in $f_{\mathcal{A}}(p)$. The cases in Fig. 6(b) and (c) can be similarly obtained. Now, one has that dependent triplet elements in $\mathcal{A}$ $\Rightarrow$ dependent angles in $f_{\mathcal{A}}(p)$, which implies that independent angles in $f_{\mathcal{A}}(p) \Rightarrow$ independent triplet elements in $\mathcal{A}$. So its angle set $\mathcal{A}$ has $2N-4$ independent triplet elements. ■

Now we show the relationship between angle rigidity and infinitesimal angle rigidity.

Theorem 4 (Relationship between infinitesimal angle rigidity and angle rigidity): If an angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is infinitesimally angle rigid, then it is angle rigid.

Proof: From Definition 9, we know that if $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is infinitesimally angle rigid, then all the continuous infinitesimally angle rigid motion $p(t)$ are trivial, which are the combination of translation, rotation, and scaling of $\mathbb{A}$. Consider another angularity $\mathbb{A}'(\mathcal{V}, \mathcal{A}, p')$ with $\varepsilon > 0$ and $\|p'-p\| < \varepsilon$, which is equivalent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$. Then, the continuous motion from p to p' maintaining $f_{\mathcal{A}}(p)$ is the combination of translation, rotation, and scaling of $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, which are angle-preserving motions, i.e., (3) holds. Therefore, $\mathbb{A}(\mathcal{V}, \mathcal{A}, p')$ is congruent to $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$, which implies that $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is angle rigid. ■

For infinitesimally angle rigid angularities, we now discuss when its number of angles in $\mathcal{A}$ becomes the minimum. Towards this end, we need to clarify what we mean by minimal angle rigidity.

Definition 10 (Minimal angle rigidity): An angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is *minimally angle rigid* if it is angle rigid and fails to remain so after removing any element in $\mathcal{A}$, and is *infinitesimally minimally angle rigid* if it is infinitesimally angle rigid and minimally angle rigid.

Since $\mathrm{Rank}[R_a(p)] \leq 2N-4$, the minimum number of angle constraints in $f_{\mathcal{A}}(p)$ to maintain infinitesimal angle rigidity is exactly $2N-4$. So we immediately have the following lemma.

Lemma 3: An angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is infinitesimally minimally angle rigid if and only if it is infinitesimally angle rigid and $|\mathcal{A}| = 2N-4$.

For an infinitesimally minimally distance rigid framework, there must exist a vertex associated with fewer than 4 distance constraints [27], [28]; otherwise, the total number of distance constraints will be at least $2N$ and thus greater than the minimum number $2N-3$. This property is critical for the success of the Henneberg construction method in order to generate an arbitrary infinitesimally minimally distance rigid framework [27], [29]. However, for an infinitesimally minimally angle rigid angularity, the situation is more challenging, which in fact prevents drawing similar conclusions as the Henneberg construction does for distance rigidity. To be more precise, we have the following lemma.

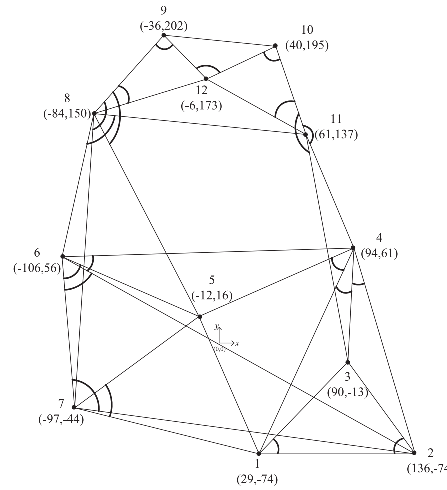

Fig. 7. All vertices are involved in 5 angle constraints in an infinitesimally minimally angle rigid angularity.

Lemma 4: For an infinitesimally minimally angle rigid angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ with $|\mathcal{A}| = 2N-4$, it must have a vertex involved in more than one but fewer than 6 angle constraints.

Proof: If every vertex is involved in at least 6 angle constraints, then the total number of angle constraints is at least $|\mathcal{A}| \geq \frac{6N}{3} = 2N$, which contradicts Lemma 3. Then, for that vertex, which has fewer than 6 angle constraints, if it is involved in only one angle constraint, then it is not infinitesimally rigid with respect to the rest of the angularity, which contradicts the property of infinitesimal angle rigidity. So there must be at least one vertex that is involved in 2, 3, 4, or 5 angle constraints. ■

In Fig. 7, we show an infinitesimally minimally angle rigid angularity whose vertices are all involved in 5 angle constraints.

Note that if an angularity $\mathbb{A}(\mathcal{V}, \mathcal{A}, p)$ is infinitesimally minimally angle rigid, then $|\mathcal{A}| = 2N-4$, and more importantly, the triplet elements in $\mathcal{A}$ need to be independent; this also implies that those situations listed in Fig. 6, namely cyclic triplets, triplets with the same middle vertex, and overly constrained angle subsets, cannot show up in $\mathcal{A}$, which is a necessary combinatorial condition for infinitesimal minimal angle rigidity. In the following section, we show how to apply the angle rigidity theory we have developed for multiagent formation control.

IV. Application in Multiagent Planar Formations

To achieve a planar formation by a group of mobile robots, many formation control algorithms have been designed, most of which require the measurement of relative positions [15], [30], [31] or aligned bearings [9], [32], or communication [15], [33]. Note that in [15], a gradient-based formation stabilization control law is designed to achieve an infinitesimally angle rigid formation, in which the measurements of relative position and

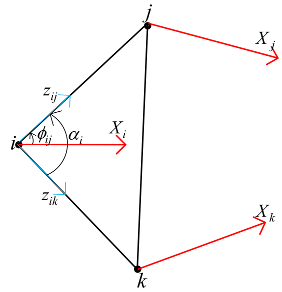

Fig. 8. Agent i's angle measurements.

wireless communication of neighbors' angle error information are both needed. In this section, we demonstrate how to stabilize a multiagent planar formation using only local angle measurements with the help of the angle rigidity theory that we have just developed.

For an agent i moving in the plane, we consider its dynamics are governed by single-integrator

$$\dot{p}_i = u_i, i = 1, \ldots, N \tag{22}$$

where $p_i \in \mathbb{R}^2$ denotes agent i's position, $u_i \in \mathbb{R}^2$ is the control input to be designed, and N is the number of agents in the group. Agent i can only measure angles; to be more specific, it can only measure the angle $\phi_{ij} \in [0, 2\pi)$ with respect to another agent j evaluated counter-clockwise from the X-axis of its own local coordinate frame of choice that is fixed to the ground.

To avoid confusion in the stability analysis, we first describe all variables in a global coordinate frame and finally we demonstrate that this global coordinate frame is unnecessary. Now we define the bearing $z_{ij} \in \mathbb{R}^2$ to be the unit vector pointing from agent i to j, i.e.,

$$z_{ij} = \frac{p_j - p_i}{\|p_j - p_i\|} = \begin{bmatrix} \cos\phi_{ij} \\ \sin\phi_{ij} \end{bmatrix} \tag{23}$$

where ϕ_{ij} determines uniquely z_{ij} when $p_i \neq p_j$. Therefore, when ϕ_{ij} can be measured, z_{ij} is known. In the triangle $\triangle ijk$ shown in Fig. 8, the interior angle α_i can be computed by

$$\alpha_i = \measuredangle kij = \arccos(z_{ij}^{\mathrm{T}} z_{ik}) \tag{24}$$

using bearings z_{ij} and z_{ik}. Note that the X-axes of agents i, j, and k do not need to align, and the angle to be controlled is not the measured angle ϕ_{ij}, but the relative angle α_i.

We construct the desired planar formation through a sequence of Type-I vertex additions (Case 3) from a generically angle rigid 3-vertex angularity, which is globally angle rigid according to Proposition 1. First, in an N-agent formation, we label the agents by 1–N. Then, agents 1, 2, and 3 aim at forming the first triangular shape, and each of agents 4–N aims at achieving two desired angles formed with other three agents (see Fig. 9). By repeatedly adding new agents through the Type-I vertex addition operation, the aim is to achieve the desired angle rigid formation specified as follows. For agents 1–3

$$\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty}(\alpha_{312}(t) - \alpha_{312}^*) = 0 \tag{25}$$

$$\lim_{t\to\infty} e_2(t) = \lim_{t\to\infty}(\alpha_{123}(t) - \alpha_{123}^*) = 0 \tag{26}$$

$$\lim_{t\to\infty} e_3(t) = \lim_{t\to\infty}(\alpha_{231}(t) - \alpha_{231}^*) = 0 \tag{27}$$

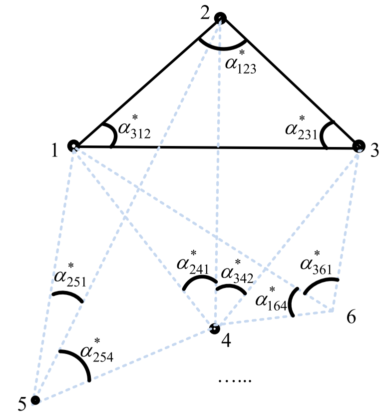

Fig. 9. Constructing desired formation by using Case 3 of Type-I vertex addition starting from $\triangle 123$.

where $\alpha_{jik}^* \in (0, \pi), i, j, k \in \{1, 2, 3\}$ denote agent i's desired angle formed with agents j, k. For agents 4–N

$$\lim_{t\to\infty} e_{i1}(t) = \lim_{t\to\infty}(\alpha_{j_1 i j_2}(t) - \alpha_{j_1 i j_2}^*) = 0 \tag{28}$$

$$\lim_{t\to\infty} e_{i2}(t) = \lim_{t\to\infty}(\alpha_{j_2 i j_3}(t) - \alpha_{j_2 i j_3}^*) = 0 \tag{29}$$

where $i = 4, \ldots, N$, $j_1 < i, j_2 < i, j_3 < i$, and $\alpha_{j_1 i j_2}^* \in (0, \pi), \alpha_{j_2 i j_3}^* \in (0, \pi)$ denote agent i's two desired angles formed with agents $j_1, j_2, j_3 \in \{1, 2, \ldots, i-1\}, j_1 \neq j_2 \neq j_3$. Therefore, the angle-only formation control problem to be solved in this section is formally described below.

Problem 1: Given feasible desired angles $f_{\mathcal{A}} = \{\alpha_{312}^*, \alpha_{123}^*, \alpha_{231}^*, \alpha_{241}^*, \alpha_{342}^*, \ldots, \alpha_{i_1 N i_2}^*, \alpha_{i_2 N i_3}^*\}$, design control law u_i by only using angle measurements ϕ_{ij} to achieve (25)–(29).

Remark 3: One may also choose other cases in Type-I and Type-II vertex addition operations to construct the desired formations. However, the constructed formations are not globally angle rigid or the realization depends on the knowledge of the neighbors' angle error, which are the drawbacks of the other cases when they are applied to formation control. For example, in Case 1 of Type-II vertex addition [Fig. 4(d)], Proposition 2 shows that the constructed formation is only angle rigid which may cause ambiguity; moreover, the angle $\alpha_{k_1 j_1 i}$ cannot be obtained by agent i's local angle measurements.

A. Triangular Formation Control for Agents 1–3

To achieve the desired angles for agents 1–3, we design their formation control laws

$$u_i = -(\alpha_i - \alpha_i^*)(z_{i(i+1)} + z_{i(i-1)}) \tag{30}$$

where $i \in \{1, 2, 3\}$, $z_{i(i+1)} = z_{31}$ when $i = 3$ and $z_{i(i-1)} = z_{13}$ when $i = 1$, and α_i represents $\alpha_{(i-1)i(i+1)}$ for conciseness.

To obtain the convergence of the angle errors, we first analyze the dynamics of the angle errors $e_i(t), i = 1, 2, 3$. Different from [34], we use the dot product of two bearings to obtain the angle error dynamics. According to (10), agent 1's angle error dynamics can be obtained by

$$\dot{\alpha}_1 = -\left[\frac{P_{z_{13}}}{l_{13}\sin\alpha_1}(\dot{p}_3 - \dot{p}_1)\right]^{\mathrm{T}} z_{12} - z_{13}^{\mathrm{T}} \frac{P_{z_{12}}}{l_{12}\sin\alpha_1}(\dot{p}_2 - \dot{p}_1). \tag{31}$$

By following the calculation in Appendix A, one has the first three agents' angle error dynamics

$$\dot{e}_f = [\dot{\alpha}_1 \quad \dot{\alpha}_2 \quad \dot{\alpha}_3]^{\mathrm{T}} = F(e_f)e_f$$
$$= \begin{bmatrix} -g_1 & f_{12} & f_{13} \\ f_{21} & -g_2 & f_{23} \\ f_{31} & f_{32} & -g_3 \end{bmatrix} \begin{bmatrix} \alpha_1 - \alpha_1^* \\ \alpha_2 - \alpha_2^* \\ \alpha_3 - \alpha_3^* \end{bmatrix} \tag{32}$$

where $e_f = [\alpha_1 - \alpha_1^* \quad \alpha_2 - \alpha_2^* \quad \alpha_3 - \alpha_3^*]^{\mathrm{T}}$, $g_i = (\sin\alpha_i)(1/l_{i(i+1)} + 1/l_{i(i-1)})$, $f_{ij} = (\sin\alpha_j)/l_{ij}$.

To guarantee that the triangular formation system under the control law (30) is well defined, we first prove that no collinearity and collision will take place under (32) if the formation is not collinear initially.

Lemma 5 (No collinearity): For the three-agent formation, if the initial formation is not collinear, it will not become collinear for $t > 0$ under the angle error dynamics (32).

Proof: Consider the manifold $\mathcal{M}_a = \{(\alpha_1, \alpha_2, \alpha_3) | \alpha_1 + \alpha_2 + \alpha_3 = \pi, 0 < \alpha_1 < \pi, 0 < \alpha_2 < \pi, \text{ and } 0 < \alpha_3 < \pi\}$ which is an open set. To show $\mathcal{M}_a$ is positively invariant, we show that for any $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{M}_a$ it is impossible for $\alpha_i, i = 1, 2, 3$ to escape $\mathcal{M}_a$ under (32). Consider the boundary states $\alpha_i(t) = \pi - \varepsilon_1$ with $\varepsilon_1 = 0^+$, $\alpha_{i+1}(t) = \varepsilon_2 = 0^+$, $\alpha_{i-1}(t) = \varepsilon_3 = 0^+$, $\varepsilon_1 = \varepsilon_2 + \varepsilon_3$.

According to (32), one has

$$\dot{e}_i = -g_i e_i + f_{i(i+1)} e_{i+1} + f_{i(i-1)} e_{i-1}. \tag{33}$$

Since $0 < \alpha_i^* < \pi$ and α_i^* is bounded away from 0 and π, one has

$$g_i e_i = g_i(\alpha_i - \alpha_i^*) > 0 \tag{34}$$
$$f_{i(i+1)} e_{i+1} = f_{i(i+1)}(\alpha_{i+1} - \alpha_{i+1}^*) < 0 \tag{35}$$
$$f_{i(i-1)} e_{i-1} = f_{i(i-1)}(\alpha_{i-1} - \alpha_{i-1}^*) < 0 \tag{36}$$

which implies that $\dot{e}_i(t) < 0$. Thus, when $\alpha_i(t)$ is close to π, $\alpha_i(t)$ will decrease, which implies that $\mathcal{M}_a$ is positively invariant, i.e., trajectories starting from $\mathcal{M}_a$ remains in $\mathcal{M}_a$. ■

Lemma 6 (No collision): For the three-agent formation, if the initial angles $\alpha_i(0) \neq 0, i = 1, 2, 3$, no collision will take place for $t > 0$ under the formation control law (30).

Proof: Suppose on the contrary that collision may happen between agents i and j at $t = t_1$. Then, one of the following two cases shown in Fig. 10 will take place.

For the first case, $\dot{p}_i(t_1) = -\gamma \dot{p}_j(t_1)$, where γ is a positive constant. Note that the moving direction of agent i under the control law (30) is always the bisector of the interior angle α_i. According to Lemma 5, no collinearity will happen for $t > 0$ which implies that $z_{ik}(t) \neq -z_{jk}(t)$ for $t > 0$. According to the control law (30), $\dot{p}_i(t_1) = -\gamma \dot{p}_j(t_1)$ requires $z_{ik}(t_1) = -z_{jk}(t_1)$, which is impossible for $t > 0$.

For the second case, since agents i and j move toward the inside of the triangle, it follows from the control law (30) that $\frac{\pi}{2} - \varepsilon_1 = \alpha_i(t_1^-) < \alpha_i^*$ and $\frac{\pi}{2} - \varepsilon_2 = \alpha_j(t_1^-) < \alpha_j^*$, where $\varepsilon_1 = 0^+$ and $\varepsilon_2 = 0^+$. Then, $\alpha_i^* + \alpha_j^* + \alpha_k^* = \pi > \pi + \alpha_k^* - \varepsilon_1 - \varepsilon_2$, which contradicts the fact that α_k^* is bounded away from 0. ■

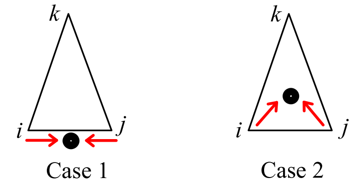

Fig. 10. Collision cases.

Now, we give the main result for the convergence of the triangular formation.

Theorem 5 (Stability of the first three agents): For the triangular formation under the control law (30), if $\alpha_i(0) \neq 0$ and the initial angle errors $e_i(0), i = 1, 2, 3$ are sufficiently small, the angle errors e_i and agents' control input $u_i(t)$ converge exponentially to zero.

Proof: From Lemmas 5 and 6, no collinearity and collision will take place since $\sin\alpha_i \neq 0, l_{ij} \neq 0, \forall i, j = 1, 2, 3$, which guarantees that the closed-loop system under the control law (30) is well defined. Since $e_1 + e_2 + e_3 = \sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \alpha_i^* \equiv 0$, the angle error dynamics (32) can be reduced to

$$\dot{e}_s = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} -(g_1 + f_{13}) & f_{12} - f_{13} \\ f_{21} - f_{23} & -(g_2 + f_{23}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = F_s(e_s)e_s. \tag{37}$$

Let $\mathbb{U}_2 \in \mathbb{R}^2$ denote a neighborhood of the origin $\{e_1 = e_2 = 0\}$, in which we investigate the local stability of (37). Linearizing (37) around the origin, we obtain

$$\dot{e}_s = L_1(\alpha^*)e_s \tag{38}$$

where $L_1(\alpha^*) = F_s(e_s)|_{e_s=0}$. Then, one has

$$\mathrm{tr}(L_1(\alpha^*)) = -g_1^* - f_{13}^* - g_2^* - f_{23}^* < 0 \tag{39}$$
$$\det(L_1(\alpha^*)) = (g_1^* + f_{13}^*)(g_2^* + f_{23}^*) - (f_{21}^* - f_{23}^*)(f_{12}^* - f_{13}^*)$$
$$> g_1^* f_{23}^* + g_2^* f_{13}^* + f_{21}^* f_{13}^* + f_{12}^* f_{23}^* > 0 \tag{40}$$

where $g_i^* = g_i|_{e_s=0}$, $f_{ij}^* = f_{ij}|_{e_s=0}$, and tr() and det() denote the trace and determinant of a square matrix, respectively, and we have used $g_1^* g_2^* > f_{21}^* f_{12}^*$. According to (39) and (40), one has that $L_1(\alpha^*)$ is Hurwitz. According to the Lyapunov Theorem [35, Theorem 4.6], there always exists positive definite matrices $P_1 \in \mathbb{R}^{2\times 2}$ and $Q_1 \in \mathbb{R}^{2\times 2}$ such that $-Q_1 = P_1 L_1(\alpha^*) + L_1^{\mathrm{T}}(\alpha^*)P_1$. Design the Lyapunov function candidate as

$$V_1 = e_s^{\mathrm{T}} P_1 e_s. \tag{41}$$

Taking the time-derivative of V_1 yields

$$\dot{V}_1 = -e_s^{\mathrm{T}} Q_1 e_s \leq -\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P_1)} V_1 \tag{42}$$

which implies that $V_1(t) \leq V_1(0) e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P_1)} t}$ where $\lambda_{\max}$ and $\lambda_{\min}$ denote the maximum and minimum eigenvalues of a real symmetric matrix, respectively. Since $P_1 > 0$, one has

$$e_1^2 + e_2^2 = \|e_s\|^2 \leq \frac{V_1}{\lambda_{\min}(P_1)} \leq \frac{V_1(0)}{\lambda_{\min}(P_1)} e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P_1)} t}. \tag{43}$$

Also, one has

$$e_3^2 = e_1^2 + e_2^2 + 2e_1 e_2 \leq 2(e_1^2 + e_2^2) \leq \frac{2V_1(0)}{\lambda_{\min}(P_1)} e^{-\frac{\lambda_{\min}(Q_1)}{\lambda_{\max}(P_1)} t}$$

which implies that e_i under the dynamics (32) is exponentially stable when the initial states lie in $\mathbb{U}_2$. According to (30), $\|u_i\| \leq 2|e_i|$ also converge to zero at an exponential rate, which implies that $p_i, i=1,2,3$ will converge to fixed points and the orientation and scale of the formation will then be fixed. ■

Remark 4: With noncollinear initial positions, the first three agents' angle error dynamics $\dot{e}_s = F_s(e_s)e_s$ are globally stable, as a consequence of the Poincare–Bendixson theorem [35, Lemma 2.1] employed in [34, Theorem 6]. The difference between the angle error dynamics $\dot{e}_s = F_s(e_s)e_s$ and the dynamics given in [34] is that $\sin\alpha_i$ shown in g_i, f_{ij} in (32) is replaced by $\sin\frac{\alpha_i}{2}$ in [34]. However, for a triangular formation, it holds that $\sin\frac{\alpha_i}{2} > 0$ and $\sin\alpha_i > 0$ for all $\alpha_i \in (0,\pi)$. Therefore, one can similarly obtain the almost global stability of $\dot{e}_s = F_s(e_s)e_s$ by following [34, Th. 6].

After proving that the first three agents converge to the desired formation, we now look at the remaining agents.

B. Adding Agents 4 to N in Sequence

In this subsection, we consider that agent $i, i=4,\ldots,N$, are added to the formation through the Type-I vertex addition operation with two desired angles. For agents $i=4,\ldots,N$, the control algorithm is designed to be

$$\begin{aligned} u_i = &-(\alpha_{j_1ij_2} - \alpha^*_{j_1ij_2})(z_{ij_1} + z_{ij_2}) \\ &-(\alpha_{j_2ij_3} - \alpha^*_{j_2ij_3})(z_{ij_2} + z_{ij_3}) \end{aligned} \tag{44}$$

where $\alpha^*_{j_1ij_2} \in (0,\pi)$ and $\alpha^*_{j_2ij_3} \in (0,\pi)$, $j_1 < i, j_2 < i, j_3 < i$, $j_1 \neq j_2 \neq j_3$ are the two desired angles. Different from the first three agents, the bearing measurement topology from agents 4 to N becomes directed, which is also similarly employed in [16].

To prove the stability from agents 4 to N, we use induction. Toward this end, we need to first prove that the 4-agent formation of 1–4 converges to the desired shape exponentially. For the 4-agent formation, the control algorithm (44) can be written as

$$\begin{aligned} u_4 = &-(\alpha_{241} - \alpha^*_{241})(z_{41} + z_{42}) \\ &-(\alpha_{342} - \alpha^*_{342})(z_{42} + z_{43}). \end{aligned} \tag{45}$$

Then, one has the following result.

Lemma 7 (Stability of agent 4): Suppose $e_i(0), i=1,2,3$ are sufficiently small and the subformation of 1, 2, and 3 is governed by (30). Under the control algorithm (45) for agent 4, if the initial distances $l_{4i}(0)$ are sufficiently bounded away from zero, the initial angle errors $e_{41}(0)$ and $e_{42}(0)$ are sufficiently small and $\alpha^*_{341} = \alpha^*_{241} + \alpha^*_{342}, \sin\alpha^*_{124} > \sin\alpha^*_{412}, \sin\alpha^*_{423} > \sin\alpha^*_{234}$, then $e_{41}(t)$ and $e_{42}(t)$ converges to zero exponentially.

Proof: To analyze the stability of the angle errors e_{41} and e_{42} under the control algorithm (45), we first calculate the angle error dynamics of e_{41} and e_{42}. According to the calculation in Appendix B, one has the following angle error dynamics:

$$\begin{aligned} \dot{e}_4 &= [\dot{\alpha}_{241} \quad \dot{\alpha}_{342}]^{\mathrm{T}} = F_4(e_4)e_4 + W(e_4)e_s \\ &= \begin{bmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{bmatrix} \begin{bmatrix} e_{41} \\ e_{42} \end{bmatrix} + \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \end{aligned} \tag{46}$$

where $j_{11} = -\frac{\sin\alpha_{241}}{l_{41}} - \frac{\sin\alpha_{241}}{l_{42}}$, $j_{22} = -\frac{\sin\alpha_{342}}{l_{43}} - \frac{\sin\alpha_{342}}{l_{42}}$, $j_{12} = -\frac{(\sin\alpha_{241})+(\sin\alpha_{341})}{l_{41}} + \frac{\sin\alpha_{342}}{l_{42}}$, $j_{21} = -\frac{(\sin\alpha_{342})+(\sin\alpha_{341})}{l_{43}} + \frac{\sin\alpha_{241}}{l_{42}}$, $w_{11} = \frac{z_{42}^{\mathrm{T}}P_{z_{41}}(z_{12}+z_{13})}{l_{41}\sin\alpha_{241}}$, $w_{12} = \frac{z_{41}^{\mathrm{T}}P_{z_{42}}(z_{21}+z_{23})}{l_{42}\sin\alpha_{241}}$, $w_{21} = -\frac{z_{42}^{\mathrm{T}}P_{z_{43}}(z_{31}+z_{32})}{l_{43}\sin\alpha_{342}}$, $w_{22} = \frac{z_{43}^{\mathrm{T}}P_{z_{42}}(z_{21}+z_{23})}{l_{42}\sin\alpha_{342}} - \frac{z_{42}^{\mathrm{T}}P_{z_{43}}(z_{31}+z_{32})}{l_{43}\sin\alpha_{342}}$.

Now, by conducting linearization towards (46) in a small neighborhood of the origin $\{e_1=0, e_2=0, e_{41}=0, e_{42}=0\}$, one has

$$\dot{e}_4 = L_2(\alpha^*)e_4 + \bar{W}e_s \tag{47}$$

where $L_2(\alpha^*) = F_4(e_4)|_{e_s=0,e_4=0}$ and $\bar{W} = W(e_4)|_{e_s=0,e_4=0}$. Then, one has

$$\operatorname{tr}(L_2(\alpha^*)) = (j_{11} + j_{22})|_{e_s=0,e_4=0} < 0 \tag{48}$$

$$\begin{aligned} &\det(L_2(\alpha^*)) \\ &= (j_{11}j_{22} - j_{12}j_{21})|_{e_s=0,e_4=0} \\ &= \frac{l^*_{41}(\sin\alpha^*_{241}\sin\alpha^*_{342} + \sin^2\alpha^*_{342} + \sin\alpha^*_{342}\sin\alpha^*_{341})}{l^*_{41}l^*_{42}l^*_{43}} \\ &\quad + \frac{l^*_{43}(\sin\alpha^*_{241}\sin\alpha^*_{342} + \sin^2\alpha^*_{241} + \sin\alpha^*_{241}\sin\alpha^*_{341})}{l^*_{42}l^*_{41}l^*_{43}} \\ &\quad - \frac{l^*_{42}(\sin\alpha^*_{241}\sin\alpha^*_{341} + \sin\alpha^*_{341}\sin\alpha^*_{342} + \sin^2\alpha^*_{341})}{l^*_{41}l^*_{42}l^*_{43}} \end{aligned} \tag{49}$$

where l^*_{ij} is the distance between agents i and j in the desired formation. Therefore, if $\det(L_2(\alpha^*)) > 0$, one has that $L_2(\alpha^*)$ is Hurwitz. By using the law of Sines, $\sin\alpha^*_{124} > \sin\alpha^*_{412}$ and $\sin\alpha^*_{423} > \sin\alpha^*_{234}$ imply $l^*_{41} > l^*_{42}$ and $l^*_{43} > l^*_{42}$, respectively. Then, one can check that $\det(L_2(\alpha^*)) > 0$ if $l^*_{41} > l^*_{42}$ and $l^*_{43} > l^*_{42}$ hold because, on the one hand

$$l^*_{43}\sin\alpha^*_{241}\sin\alpha^*_{341} > l^*_{42}\sin\alpha^*_{241}\sin\alpha^*_{341} \tag{50}$$

$$l^*_{41}\sin\alpha^*_{341}\sin\alpha^*_{342} > l^*_{42}\sin\alpha^*_{341}\sin\alpha^*_{342} \tag{51}$$

and, on the other hand

$$\begin{aligned} \sin^2\alpha^*_{341} &= [\sin\alpha^*_{241}\cos\alpha^*_{342} + \cos\alpha^*_{241}\sin\alpha^*_{342}]^2 \\ &= \sin^2\alpha^*_{241}\cos^2\alpha^*_{342} + \cos^2\alpha^*_{241}\sin^2\alpha^*_{342} \\ &\quad + 2\sin\alpha^*_{241}\cos\alpha^*_{342}\cos\alpha^*_{241}\sin\alpha^*_{342} \end{aligned} \tag{52}$$

and $l^*_{41}\sin^2\alpha^*_{342} > l^*_{42}\sin^2\alpha^*_{342}\cos^2\alpha^*_{241}$, $l^*_{43}\sin^2\alpha^*_{241} > l^*_{42}\sin^2\alpha^*_{241}\cos^2\alpha^*_{342}$ and $l^*_{41}\sin\alpha^*_{241}\sin\alpha^*_{342} + l^*_{43}\sin\alpha^*_{241}\sin\alpha^*_{342} > 2l^*_{42}\sin\alpha^*_{241}\sin\alpha^*_{342} > 2l^*_{42}\sin\alpha^*_{241}\cos\alpha^*_{342}\cos\alpha^*_{241}\sin\alpha^*_{342}$. By combining (38) and (47) together, one has the overall linearized 4-agent angle error dynamics

$$\dot{\bar{e}}_4 = \begin{bmatrix} \dot{e}_s \\ \dot{e}_4 \end{bmatrix} = L_4(\alpha^*)\bar{e}_4 = \begin{bmatrix} L_1(\alpha^*) & 0 \\ \bar{W} & L_2(\alpha^*) \end{bmatrix} \begin{bmatrix} e_s \\ e_4 \end{bmatrix} \tag{53}$$

When $L_1(\alpha^*)$ and $L_2(\alpha^*)$ are Hurwitz, one has that $L_4(\alpha^*)$ is also Hurwitz. When $L_4(\alpha^*)$ is Hurwitz, for an arbitrary positive definite matrix $Q_2 \in \mathbb{R}^{4\times 4}$, there always exists positive definite matrix $P_2 \in \mathbb{R}^{4\times 4}$ such that $-Q_2 = P_2L_4(\alpha^*) + L_4^{\mathrm{T}}(\alpha^*)P_2$. Design the Lyapunov function candidate as

$$V_2 = \bar{e}_4^{\mathrm{T}}P_2\bar{e}_4. \tag{54}$$

Taking the time-derivative of V_2 along (53) yields

$$\dot{V}_2 = -\bar{e}_4^{\mathrm{T}} Q_2 \bar{e}_4 \leq -\lambda_{\min}(Q_2)\|\bar{e}_4\|^2 \leq -\frac{\lambda_{\min}(Q_2)}{\lambda_{\max}(P_2)} V_2. \quad (55)$$

Then, one has

$$\|e_4\|^2 \leq \|\bar{e}_4\|^2 \leq \frac{V_2}{\lambda_{\min}(P_2)} \leq \frac{V_2(0)}{\lambda_{\min}(P_2)} e^{-(\frac{\lambda_{\min}(Q_2)}{\lambda_{\max}(P_2)})t}. \quad (56)$$

which implies that the agent 4's angle error e_4 also converges to zero at an exponential rate. To guarantee that $\|W(e_4)\|$ is bounded and control law (45) is well defined, the collision between agent 4 and agents 1–3 should be avoided. Taking agent 1 as an example, one has

$$\begin{aligned} &\|p_4(t) - p_1(t)\| \\ &= \|p_4(0) + \int_0^t u_4(s)\mathrm{d}s - p_1(0) - \int_0^t u_1(s)\mathrm{d}s\| \\ &\geq \|p_4(0) - p_1(0)\| - \int_0^t \|u_1(s) - u_4(s)\|\mathrm{d}s \\ &\geq l_{14}(0) - 2\int_0^t (|e_1(s)| + |e_{41}(s)| + |e_{42}(s)|)\mathrm{d}s. \end{aligned}$$

Since $l_{14}(0)$ is sufficiently bounded away from zero, there always exists a finite time T such that in the time interval $[0, T]$, there is no collision between agent 4 and agent 1. Then, according to (43) and (56), one has

$$\begin{aligned} &\|p_4(T) - p_1(T)\| \\ &\geq l_{14}(0) - 2\int_0^T (|e_1(s)| + |e_{41}(s)| + |e_{42}(s)|)\mathrm{d}s \\ &\geq l_{14}(0) - 4\Bigg[\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(Q_1)}\sqrt{\frac{V_1(0)}{\lambda_{\min}(P_1)}}(1 - e^{-\frac{\lambda_{\min}(Q_1)}{2\lambda_{\max}(P_1)}T}) \\ &\quad + \frac{\lambda_{\max}(P_2)}{\lambda_{\min}(Q_2)}\sqrt{\frac{2V_2(0)}{\lambda_{\min}(P_2)}}(1 - e^{-(\frac{\lambda_{\min}(Q_2)}{2\lambda_{\max}(P_2)})T})\Bigg] \end{aligned} \quad (57)$$

where we have used the fact that $|e_{41}| + |e_{42}| \leq \sqrt{2(e_{41}^2 + e_{42}^2)} = \sqrt{2}\|e_4\|$. Since $V_1(0)$ and $V_2(0)$ are sufficiently small and $l_{14}(0)$ is sufficiently bounded away from zero, one has $\|p_4(T) - p_1(T)\| > 0$ since $l_{14}(0) > 4[\frac{\lambda_{\max}(P_1)}{\lambda_{\min}(Q_1)}\sqrt{\frac{V_1(0)}{\lambda_{\min}(P_1)}} + \frac{\lambda_{\max}(P_2)}{\lambda_{\min}(Q_2)}\sqrt{\frac{2V_2(0)}{\lambda_{\min}(P_2)}}]$. Then, we extend T to infinity. Because $e^{-\frac{\lambda_{\min}(Q_1)}{2\lambda_{\max}(P_1)}t} > 0$ and $e^{-(\frac{\lambda_{\min}(Q_2)}{2\lambda_{\max}(P_2)})t} > 0, \forall t > 0$, one has that $l_{41}(t) = \|p_4(t) - p_1(t)\| > 0$ for $t > 0$. On the other hand, since the initial angle errors $e_{41}(0)$ and $e_{42}(0)$ are sufficiently small and $e_1(t)$, $e_2(t)$, $e_{41}(t)$ and $e_{42}(t)$ converge at an exponential speed, $\alpha_{241}(t)$ and $\alpha_{342}(t)$ will be bounded away from 0 and π. Therefore, $\|W(e_4)\|$ is bounded and (46) is well defined. The proof for 4-agent formation is completed. ■

Now, we present the main result for agents 4 to N.

Theorem 6 (Stability of all the agents): Consider a formation of $N > 3$ agents, each of which is governed by (22). Suppose $e_i(0), i = 1, 2, 3$ are sufficiently small and the subformation of 1, 2, 3 is governed by (30). For agent $i, 4 \leq i \leq N$, if the initial distances $l_{ij_1}(0)$, $l_{ij_2}(0)$, and $l_{ij_3}(0)$ are sufficiently bounded away from zero, the initial angle errors $e_{i1}(0)$ and $e_{i2}(0)$ are sufficiently small and $\alpha^*_{j_3ij_1} = \alpha^*_{j_2ij_1} + \alpha^*_{j_3ij_2}, \sin\alpha^*_{j_1j_2i} > \sin\alpha^*_{ij_1j_2}, \sin\alpha^*_{ij_2j_3} > \sin\alpha^*_{j_2j_3i}$, then under (44), the formation achieves its desired shape exponentially.

Proof: From Lemma 7, 4-agent formation achieves the desired shape exponentially. Suppose for a $4 < k < N$, the k-agent formation converges to the desired shape exponentially. We need to prove that for $(k+1)$-agent formation, the relative angle errors $e_{(k+1)1} = \alpha_{j_1(k+1)j_2} - \alpha^*_{j_1(k+1)j_2}$ and $e_{(k+1)2} = \alpha_{j_2(k+1)j_3} - \alpha^*_{j_2(k+1)j_3}$ converge to zero exponentially. Similar to the proof from (45) to (55), one has that the angle errors $e_{(k+1)1}$ and $e_{(k+1)2}$ exponentially converge to zero. Therefore, the control algorithm (44) can locally stabilize the agent $k+1$, i.e., the $(k+1)$-agent formation converges to the desired shape exponentially. So, from induction, N-agent formation converges to the desired formation shape exponentially. The proof for Theorem 6 is completed. ■

Remark 5: Note that the control laws (30) and (44) can be described by a unified form

$$u_i = -\sum\nolimits_{(j,i,k)\in\mathcal{A}} (\alpha_{jik} - \alpha^*_{jik})(z_{ij} + z_{ik}) \quad (58)$$

where $\mathcal{A} = \{(1,2,3), (2,3,1), (3,1,2), (1,4,2), (2,4,3), \ldots, (j_1, k, j_2), (j_2, k, j_3), \ldots, (i_1, N, i_2), (i_2, N, i_3)\}, j_1 < k, j_2 < k, j_3 < k, j_1 \neq j_2 \neq j_3$. Therefore, the unified control algorithm (58) can locally stabilize the angle rigid formation constructed through a sequence of Type-I vertex additions (Case 3) from a triangular shape. Because we aim at obtaining local stability for multiagent formations in Section IV, we only consider the range of the desired angles belonging to $(0, \pi)$ in (25)–(29), and the case of $\alpha_i(0) \in (\pi, 2\pi), \alpha^*_i \in (\pi, 2\pi)$ can be similarly obtained. However, to achieve a general infinitesimally and minimally angle rigid formation, one can use the gradient-based control law

$$\dot{p} = u = -\left(\frac{\partial V_3}{\partial p}\right)^{\mathrm{T}} = -R_a^{\mathrm{T}}(p)(\alpha - \alpha^*) \quad (59)$$

where $V_3 = 0.5(\alpha - \alpha^*)^{\mathrm{T}}(\alpha - \alpha^*)$, p, u, α are the stack vectors of p_i, u_i, α_{jik}, respectively. It follows that $\dot{V}_3 = -(\alpha - \alpha^*)^{\mathrm{T}} R_a(p) R_a^{\mathrm{T}}(p)(\alpha - \alpha^*)$. Because $R_a(p)R_a^{\mathrm{T}}(p)$ is positive definite when p is in a small neighborhood of the desired formation, one has the local convergence of $(\alpha - \alpha^*)$.

Remark 6: Although each agent's position in (22) is described in the global coordinate frame, it is not required in the implementation of control algorithm (58). The control algorithm (58) can be realized in each agent's local coordinate frame since (58) can be equivalently written in agent i's local coordinate frame

$$R_g^b u_i = -\sum\nolimits_{(j,i,k)\in\mathcal{A}} (\alpha_{jik} - \alpha^*_{jik}) R_g^b (z_{ij} + z_{ik}) \quad (60)$$

where $R_g^b \in SO(2)$ is the rotation matrix from the global coordinate frame to agent i's local coordinate frame, $R_g^b u_i$ is the controller input in agent i's local coordinate frame, and $R_g^b z_{ij}, R_g^b z_{ik}$ are the local bearings measured in agent i's local coordinate frame. Since $(\alpha_{jik} - \alpha^*_{jik})$ is a scalar and α_{jik} is the same under different coordinate frames, (60) and (58) are equivalent.

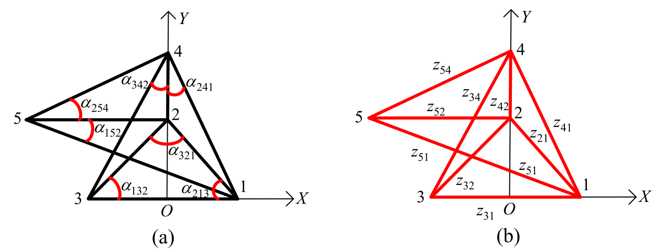

Fig. 11. Desired planar formation. (a) Angle-based. (b) Bearing-based.

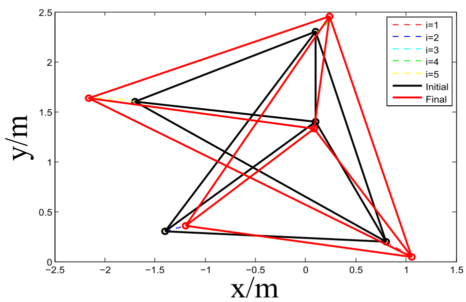

Fig. 12. Formation trajectories under angle rigidity-based control law with misalignment in agents' coordinate frames.

V. Simulation Examples

In this section, we first provide a simulation example to validate the effectiveness of the proposed angle rigidity-based control law (58). Then, we compare the angle rigidity-based formation control law with bearing rigidity-based formation control law. To begin with, we give the desired formation shape in Fig. 11.

A. Angle Rigidity-Based Control Law

Consider five agents in the plane with the following initial positions:

$$p_1(0) = [0.8, 0.2]^{\mathrm{T}}, p_2(0) = [0.1, 1.4]^{\mathrm{T}}, p_3(0) = [-1.4, 0.3]^{\mathrm{T}},$$
$$p_4(0) = [0.1, 2.3]^{\mathrm{T}}, p_5(0) = [-1.7, 1.6]^{\mathrm{T}}$$

which are also used for other simulation examples. According to the form of $\mathcal{A}$ in (58), we consider the desired angles shown in Fig. 11(a) as

$$\alpha^*_{213} = \pi/4, \alpha^*_{132} = \pi/4, \alpha^*_{321} = \pi/2, \alpha^*_{342} = \arctan(0.5)$$
$$\alpha^*_{241} = \arctan(\frac{1}{2}), \alpha^*_{254} = \arctan(\frac{1}{2}), \alpha^*_{152} = \arctan(\frac{3}{\sqrt{10}})$$

which leads to a globally infinitesimally angle rigid formation according to Proposition 1 and Theorem 2. To demonstrate the coordinate-independent property illustrated in Remark 6, we introduce a misalignment $\theta_1 = 5^\circ$ in agent 1's coordinate frame $R_1(\theta) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$, and the other agents' coordinate frames are the same as the XOY shown in Fig. 11.

Under the control law (58), the simulation results are given in Figs. 12 and 13.

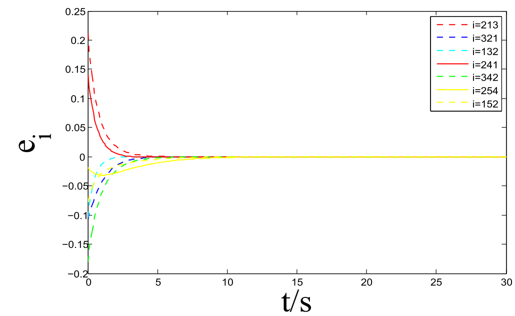

Fig. 13. Angle errors under angle rigidity-based control law with misalignment in agents' coordinate frames.

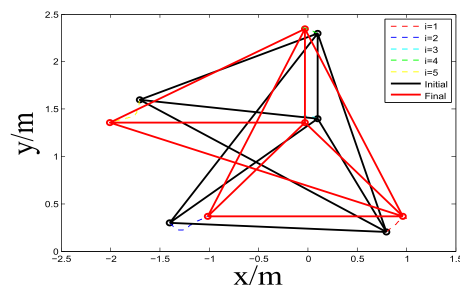

Fig. 14. Formation trajectories under bearing-based control without misalignment.

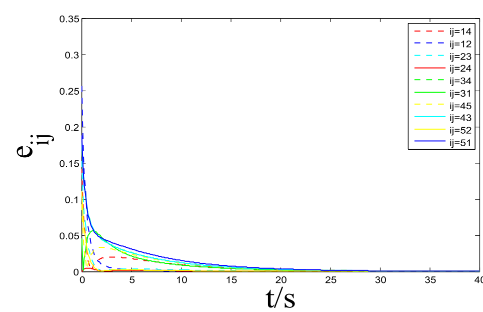

Fig. 15. Bearing errors under bearing-based control without misalignment.

B. Bearing Rigidity-Based Control Law

According to [9], a bearing rigidity-based control law is described by

$$\dot{p}_i = -\sum\nolimits_{j\in\mathcal{N}_i} P_{z_{ij}} z^*_{ij} \tag{61}$$

where the desired bearing constraints in this simulation are defined as

$$z^*_{31} = [1, 0]^{\mathrm{T}}, z^*_{21} = [\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}]^{\mathrm{T}}, z^*_{32} = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^{\mathrm{T}}$$
$$z^*_{42} = [0, -1]^{\mathrm{T}} z^*_{41} = [\frac{\sqrt{5}}{5}, \frac{-2\sqrt{5}}{5}]^{\mathrm{T}}, z^*_{43} = [\frac{-\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}]^{\mathrm{T}}$$
$$z^*_{54} = [\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}]^{\mathrm{T}}, z^*_{52} = [1, 0]^{\mathrm{T}}, z^*_{51} = [\frac{3\sqrt{10}}{10}, \frac{-\sqrt{10}}{10}]^{\mathrm{T}}.$$

Then, we introduce the same misalignment into agent 1's coordinate frame. By defining $\|z_{ij} - z^*_{ij}\|$ as bearing error, the simulation results are given in Figs. 14–17.

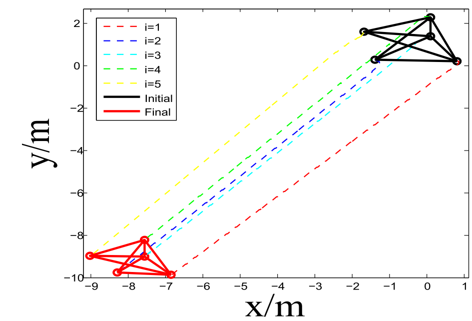

Fig. 16. Formation trajectories under bearing-based control with misalignment in agents' coordinate frames.

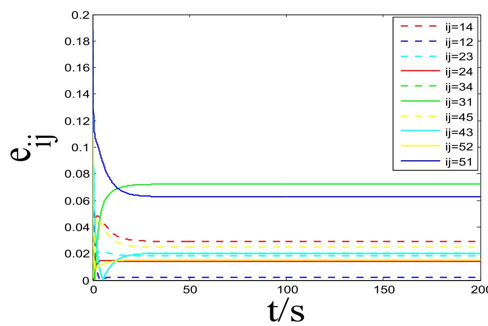

Fig. 17. Bearing errors under bearing-based control with misalignment in agents' coordinate frames.

According to the above simulation results, one has that the angle rigidity-based formation control algorithms do not require the alignment of all agents' coordinate frames, while bearing rigidity-based control law in [9] does.

VI. Conclusion and Discussion

A. Conclusion

In this article, we have proposed the angle rigidity theory for the stabilization of planar formations. The notion of angularity has been first defined to describe the multipoint framework with angle constraints. The established angle rigidity has shown to be a local property because of the existence of flex ambiguity. The infinitesimal angle rigidity has been developed based on the trivial motions of the angularity. A sufficient and necessary condition for infinitesimal angle rigidity has been investigated by checking the rank of the angle rigidity matrix. Based on the developed angle rigidity theory, we have also demonstrated how to stabilize a multiagent planar formation using only angle measurements, which can be realized in each agent's local coordinate frame. The exponential convergent rate of angle errors has also been proved. Future work will focus on the necessary and sufficient conditions for global angle rigidity and the combinatorial necessary and sufficient conditions for infinitesimal minimal angle rigidity. For the angle-only formation control, we are also interested in designing effective laws to ensure global stability.

B. Discussion

This article has investigated angle rigidity in 2-D by using signed angle constraints, which eliminates the flip ambiguity and some flex ambiguity, allowing the defined vertex-addition operations to be used to construct an angle rigid or globally angle rigid angularity. One may suggest to replace signed angles in 2-D by signed volumes in 3-D, in order to eliminate possible flip ambiguity. However, this turns out not to be the case. Therefore, angle rigidity cannot be straightforwardly extended from 2-D to 3-D and this is a future research.

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Appendix A

In view of (30), it follows

$$\begin{aligned}\dot{z}_{12} &= \frac{P_{z_{12}}}{l_{12}}(u_2 - u_1)\\ &= \frac{P_{z_{12}}}{l_{12}}[-(\alpha_2-\alpha_2^*)(z_{23}+z_{21}) + (\alpha_1-\alpha_1^*)(z_{13}+z_{12})].\end{aligned} \tag{62}$$

So

$$\begin{aligned}&\dot{z}_{12}^{\mathrm{T}} z_{13}\\ &= [(\alpha_1-\alpha_1^*)(z_{13}+z_{12}) - (\alpha_2-\alpha_2^*)(z_{23}+z_{21})]^{\mathrm{T}} \frac{P_{z_{12}}}{l_{12}} z_{13}\\ &= \frac{(\sin^2\alpha_1)(\alpha_1-\alpha_1^*) - (\cos\alpha_3 + \cos\alpha_1\cos\alpha_2)(\alpha_2-\alpha_2^*)}{l_{12}}.\end{aligned} \tag{63}$$

Since

$$\begin{aligned}\cos\alpha_3 + \cos\alpha_1\cos\alpha_2 &= -\cos(\alpha_1+\alpha_2) + \cos\alpha_1\cos\alpha_2\\ &= \sin\alpha_2 \sin\alpha_1\end{aligned} \tag{64}$$

it follows

$$\dot{z}_{12}^{\mathrm{T}} z_{13} = \frac{\sin\alpha_1}{l_{12}}[(\alpha_1-\alpha_1^*)(\sin\alpha_1) - (\alpha_2-\alpha_2^*)\sin\alpha_2].$$

Similarly, one gets

$$z_{12}^{\mathrm{T}} \dot{z}_{13} = \frac{\sin\alpha_1}{l_{13}}[(\alpha_1-\alpha_1^*)(\sin\alpha_1) - (\alpha_3-\alpha_3^*)\sin\alpha_3].$$

By using (31), agent 1's closed-loop angle dynamics are

$$\begin{aligned}\dot{\alpha}_1 =& -(\sin\alpha_1)\left(\frac{1}{l_{12}} + \frac{1}{l_{13}}\right)(\alpha_1-\alpha_1^*)\\ &+ \frac{\sin\alpha_2}{l_{12}}(\alpha_2-\alpha_2^*) + \frac{\sin\alpha_3}{l_{13}}(\alpha_3-\alpha_3^*).\end{aligned} \tag{65}$$

Similarly,

$$\begin{aligned}\dot{\alpha}_2 =& -(\sin\alpha_2)\left(\frac{1}{l_{21}} + \frac{1}{l_{23}}\right)(\alpha_2-\alpha_2^*)\\ &+ \frac{\sin\alpha_1}{l_{21}}(\alpha_1-\alpha_1^*) + \frac{\sin\alpha_3}{l_{23}}(\alpha_3-\alpha_3^*)\end{aligned} \tag{66}$$

$$\begin{aligned}\dot{\alpha}_3 =& -(\sin\alpha_3)\left(\frac{1}{l_{31}} + \frac{1}{l_{32}}\right)(\alpha_3-\alpha_3^*)\\ &+ \frac{\sin\alpha_1}{l_{31}}(\alpha_1-\alpha_1^*) + \frac{\sin\alpha_2}{l_{32}}(\alpha_2-\alpha_2^*).\end{aligned} \tag{67}$$

Writing (65) and (66) into a compact form, one has the closed-loop triangular formation dynamics given in (32).

Appendix B

Since

$$\frac{\mathrm{d}(\cos\alpha_{241})}{\mathrm{d}t} = -(\sin\alpha_{241})\dot{\alpha}_{241} = \frac{\mathrm{d}(z_{41}^{\mathrm{T}} z_{42})}{\mathrm{d}t} = (\dot{z}_{41})^{\mathrm{T}} z_{42} + (z_{41})^{\mathrm{T}}\dot{z}_{42} \quad (68)$$

and similarly

$$\dot{z}_{41} = \frac{P_{z_{41}}}{l_{41}}(\dot{p}_1 - \dot{p}_4) = \frac{P_{z_{41}}}{l_{41}} u_1 - \frac{P_{z_{41}}}{l_{41}} u_4 \quad (69)$$

we have

$$\begin{aligned}
&(\dot{z}_{41})^{\mathrm{T}} z_{42} \\
&= -\frac{u_4^{\mathrm{T}}}{l_{41}}(I_2 - z_{41}z_{41}^{\mathrm{T}})z_{42} + u_1^{\mathrm{T}}\frac{P_{z_{41}}}{l_{41}} z_{42} \\
&= -\frac{[(\alpha_{241}-\alpha_{241}^*)(\cos\alpha_{241} + \cos^2\alpha_{241})]}{l_{41}} \\
&\quad - \frac{[(\alpha_{342}-\alpha_{342}^*)(\cos^2\alpha_{241} + \cos\alpha_{241}\cos\alpha_{341})]}{l_{41}} \\
&\quad + \frac{[(\alpha_{241}-\alpha_{241}^*)(\cos\alpha_{241} + 1)]}{l_{41}} \\
&\quad + \frac{[(\alpha_{342}-\alpha_{342}^*)(1 + \cos\alpha_{342})]}{l_{41}} - z_{42}^{\mathrm{T}}\frac{P_{z_{41}}}{l_{41}}(z_{12}+z_{13})e_1 \\
&= \frac{(\alpha_{241}-\alpha_{241}^*)\sin^2\alpha_{241}}{l_{41}} - z_{42}^{\mathrm{T}}\frac{P_{z_{41}}}{l_{41}}(z_{12}+z_{13})e_1 \\
&\quad + \frac{(\alpha_{342}-\alpha_{342}^*)(\sin^2\alpha_{241} + \sin^2\alpha_{241}\cos\alpha_{342})}{l_{41}} \\
&\quad + \frac{(\alpha_{342}-\alpha_{342}^*)\cos\alpha_{241}\sin\alpha_{241}\sin\alpha_{342}}{l_{41}} \quad (70)
\end{aligned}$$

and

$$\begin{aligned}
z_{41}^{\mathrm{T}}\dot{z}_{42} &= z_{41}^{\mathrm{T}}\frac{P_{z_{42}}}{l_{42}} u_2 - z_{41}^{\mathrm{T}}\frac{I_2 - z_{42}z_{42}^{\mathrm{T}}}{l_{42}} u_4 \\
&= -z_{41}^{\mathrm{T}}\frac{P_{z_{42}}}{l_{42}}(z_{21}+z_{23})e_2 + \frac{(\alpha_{241}-\alpha_{241}^*)\sin^2\alpha_{241}}{l_{42}} \\
&\quad + \frac{(\alpha_{342}-\alpha_{342}^*)(-\sin\alpha_{241}\sin\alpha_{342})}{l_{42}}. \quad (71)
\end{aligned}$$

Then, from (68), it follows:

$$\begin{aligned}
\dot{\alpha}_{241} &= -\frac{1}{\sin\alpha_{241}}\frac{\mathrm{d}(\cos\alpha_{241})}{\mathrm{d}t} = -\frac{\dot{z}_{41}^{\mathrm{T}} z_{42} + z_{41}^{\mathrm{T}}\dot{z}_{42}}{\sin\alpha_{241}} \\
&= -(\sin\alpha_{241})\left(\frac{1}{l_{41}} + \frac{1}{l_{42}}\right)(\alpha_{241}-\alpha_{241}^*) \\
&\quad - \frac{(\alpha_{342}-\alpha_{342}^*)(\sin\alpha_{241} + \sin\alpha_{341})}{l_{41}} \\
&\quad + \frac{(\alpha_{342}-\alpha_{342}^*)\sin\alpha_{342}}{l_{42}} + \frac{z_{41}^{\mathrm{T}}P_{z_{42}}(z_{21}+z_{23})}{l_{42}\sin\alpha_{241}} e_2 \\
&\quad + \frac{z_{42}^{\mathrm{T}}P_{z_{41}}(z_{12}+z_{13})}{l_{41}\sin\alpha_{241}} e_1. \quad (72)
\end{aligned}$$

Analogously

$$\begin{aligned}
\dot{\alpha}_{342} &= -\frac{1}{\sin\alpha_{342}}\frac{\mathrm{d}(\cos\alpha_{342})}{\mathrm{d}t} = -\frac{\dot{z}_{42}^{\mathrm{T}} z_{43} + z_{42}^{\mathrm{T}}\dot{z}_{43}}{\sin\alpha_{342}} \\
&= -(\sin\alpha_{342})\left(\frac{1}{l_{43}} + \frac{1}{l_{42}}\right)(\alpha_{342}-\alpha_{342}^*) \\
&\quad - \frac{(\alpha_{241}-\alpha_{241}^*)(\sin\alpha_{342} + \sin\alpha_{341})}{l_{43}} \\
&\quad + \frac{(\alpha_{241}-\alpha_{241}^*)\sin\alpha_{241}}{l_{42}} + \frac{z_{43}^{\mathrm{T}}P_{z_{42}}(z_{21}+z_{23})}{l_{42}\sin\alpha_{342}} e_2 \\
&\quad - \frac{z_{42}^{\mathrm{T}}P_{z_{43}}(z_{31}+z_{32})}{l_{43}\sin\alpha_{342}}(e_1+e_2). \quad (73)
\end{aligned}$$

By combining (72) and (73), one has the compact form (46).

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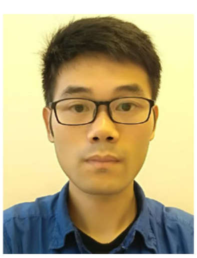

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