ASSIGNMENT 3

Due date Monday 30th of March

(1) Question 15 page 67 of the textbook
(2) Suppose f is a holomorphic function with zeros at the points $\{z_i\}$, for which each zero has order 1. Let γ be a simple curve not passing through any of the zeros of f.
  - Show that
$$\Sigma w(\gamma, z_i) = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f} dz$$
where $w(\gamma, z_i)$ is the winding number of γ around z_i.
  - Argue that the sum on the left hand side of the equation has only finitely many nonzero terms. Hint: γ is simple, so it bounds a compact subset of $\mathbb{C}$.
  - Relate the integral in the equation to the number of zeros enclosed by γ.

(3) Show that the zeros of $cos(z)$ and $sin(z)$ all lie on the real axis.
(4) Suppose f is holomorphic. Find a condition that guarantees that f is 1-1 on an open set containing z_0. Find the radius of the largest disk centered on the origin on which $z^2 - z$ is 1-1.
(5) Sometimes the boundary of the disk of convergence of a power series is so densely packed with singularities that it becomes a barrier beyond which no analytic continuation is possible. This is called a natural boundary. Consider the power series
$$f(z) = z + z^2 + z^4 + z^8 + z^{16} + \dots$$
This power series converges on the interior of the unit circle. Show that every point on the boundary of the unit circle is either a singularity, or a limit point of singularities. Hint: f(1) is singular. Note that $f(z) = z + f(z^2)$. What can you say about f on the $2^n th$ roots of unity?