SOLUTIONS TO PROBLEMS NOT WRITTEN DOWN IN LECTURE 1

Solve $z^2 = a + ib$

i) Without polar coordinates

ii) With polar coordinates.

Solution:

i) Let $z = x + iy$. Then $z^2 = x^2 - y^2 + i2xy = a + ib$

Equating real and imaginary parts gives

$$x^2 - y^2 = a$$

$$2xy = b$$

Solving for x and y gives

$$x^2 = \frac{a \pm \sqrt{a^2 + b^2}}{2}$$

However, since x is real,

$$x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$$

$$y = \frac{\pm b}{2\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}}$$

Where it is understood that the sign for y comes from the sign from x, i.e.

$$z = \pm\left(\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + \frac{ib}{2\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}}\right)$$

ii)

$$|z| = (a^2 + b^2)^{\frac{1}{4}}$$

and

$$arg(z) = \frac{arctan\left(\frac{b}{a}\right)}{2}$$

or

$$arg(z) = \frac{arctan\left(\frac{b}{a}\right)}{2} + \pi$$

This gives

$$z = (a^2 + b^2)^{\frac{1}{4}} e^{i\frac{arctan\left(\frac{b}{a}\right)}{2}}$$

or

$$z = (a^2 + b^2)^{\frac{1}{4}} e^{i\frac{arctan\left(\frac{b}{a}\right)}{2} + i\pi}$$

Solve

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0$$

Completing the square gives:

$$\left(z + \frac{\alpha + i\beta}{2}\right)^2 = \frac{\alpha^2 - \beta^2}{4} - \gamma + i\left(\frac{\alpha\beta}{2} - \delta\right)$$

so solutions are

$$\left(\left(\frac{\alpha^2 + \beta^2}{2} - \gamma\right)^2 + \left(\frac{\alpha\beta}{2} - \delta\right)^2\right)^{\frac{1}{4}} e^{\frac{i}{2} arctan\left(\frac{2\alpha\beta - 4\delta}{\alpha^2 - \beta^2 - 4\gamma}\right)}$$

or

$$\left(\left(\frac{\alpha^2 + \beta^2}{2} - \gamma\right)^2 + \left(\frac{\alpha\beta}{2} - \delta\right)^2\right)^{\frac{1}{4}} e^{\frac{i}{2} arctan\left(\frac{2\alpha\beta - 4\delta}{\alpha^2 - \beta^2 - 4\gamma}\right) + i\pi}$$

Let z and w be two complex numbers for which $\bar{z}w \neq 1$, and let $F(z) = \frac{w-z}{1-\bar{w}z}$.
Show that, for $|w| < 1$
i) F maps the interior of the unit disk to itself
ii)F maps the boundary of the unit disk to itself
iii)F is bijective
iv)$F(0) = w$ and $F(w) = 0$.

Solution:
i) Since the modulus is never negative,

$$\left|\frac{w - z}{1 - \bar{w}z}\right| < 1 \Leftrightarrow \left|\frac{w - z}{1 - \bar{w}z}\right|^2 < 1$$

However

$$\begin{aligned}\left|\frac{w - z}{1 - \bar{w}z}\right|^2 < 1 &\Leftrightarrow (w - z)(\bar{w} - \bar{z}) < (1 - \bar{w}z)(1 - w\bar{z})\\ &\Leftrightarrow w\bar{w} - z\bar{w} - \bar{z}w + z\bar{z} < 1 - \bar{w}z - w\bar{z} + w\bar{w}z\bar{z}\\ &\Leftrightarrow w\bar{w} + z\bar{z} < 1 + w\bar{w}z\bar{z}\\ &\Leftrightarrow w\bar{w} - w\bar{w}z\bar{z} < 1 - z\bar{z}\\ &\Leftrightarrow w\bar{w}(1 - z\bar{z}) < 1 - z\bar{z}\end{aligned}$$

which is true since $w\bar{w} < 1$ by assumption. Note that these inequalities make sense because we only have real numbers on either side of $<$.

ii)In the previous question, we see that equality is achieved when $z\bar{z} = 1$.

iii)This function is actually its own inverse! To check this, substitute $\frac{w-z}{1-\bar{w}z}$ in place of z in the expression for $F(z)$ and simplify.

iv)This can be easily checked

Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0$ be a polynomial with real coefficients. Show that the sum of the roots is real.

Solution:

$$a_n z^n + a_{n-1}z^{n-1} + \ldots + a_0 = 0 \Leftrightarrow \bar{a}_n \bar{z}^n + \bar{a}_{n-1}\bar{z}^{n-1} + \ldots + \bar{a}_0 = 0$$

$$\Leftrightarrow a_n \bar{z}^n + a_{n-1}\bar{z}^{n-1} + \ldots + a_0 = 0$$

Since the coefficients are assumed real. This means that if z^* is a root, so is its complex conjugate. Therefore the sum of the roots is real. You might have noticed this phenomenon when finding roots of unity.

Show that $|zw| = |z||w|$ and use trigonometric identities to show that $arg(zw) = arg(z) + arg(w)$

Solution:
Let $z = x+iy$ and $w = u+iv$. We will show that the product of the squares of the moduli is equal to the square of the modulus of the product. Since the modulus of a complex number is never negative, this implies the result.
$|zw|^2 = |xu - yv + i(uy + xv)|^2 = (xu - yv)^2 + (uy + xv)^2 = x^2u^2 + y^2u^2 + x^2v^2 = x^2(u^2 + v^2) + y^2(u^2 + v^2) = (x^2 + y^2)(u^2 + v^2) = |z|^2|w|^2$

After proving the first part, we can assume for simplicity that $|z| = |w| = 1$. So let $arg(z) = \alpha$ and $arg(w) = \beta$.
Then $x = \cos(\alpha)$, $y = \sin(\alpha)$, $u = \cos(\beta)$ and $v = \sin(\beta)$.

So $\cos(arg(zw)) = Re(zw) = xu - yv = \cos(\alpha)\sin(\alpha) - \sin(\alpha) sin(\beta) = \cos(\alpha + \beta)$.
Similarly for $\sin(arg(zw))$.